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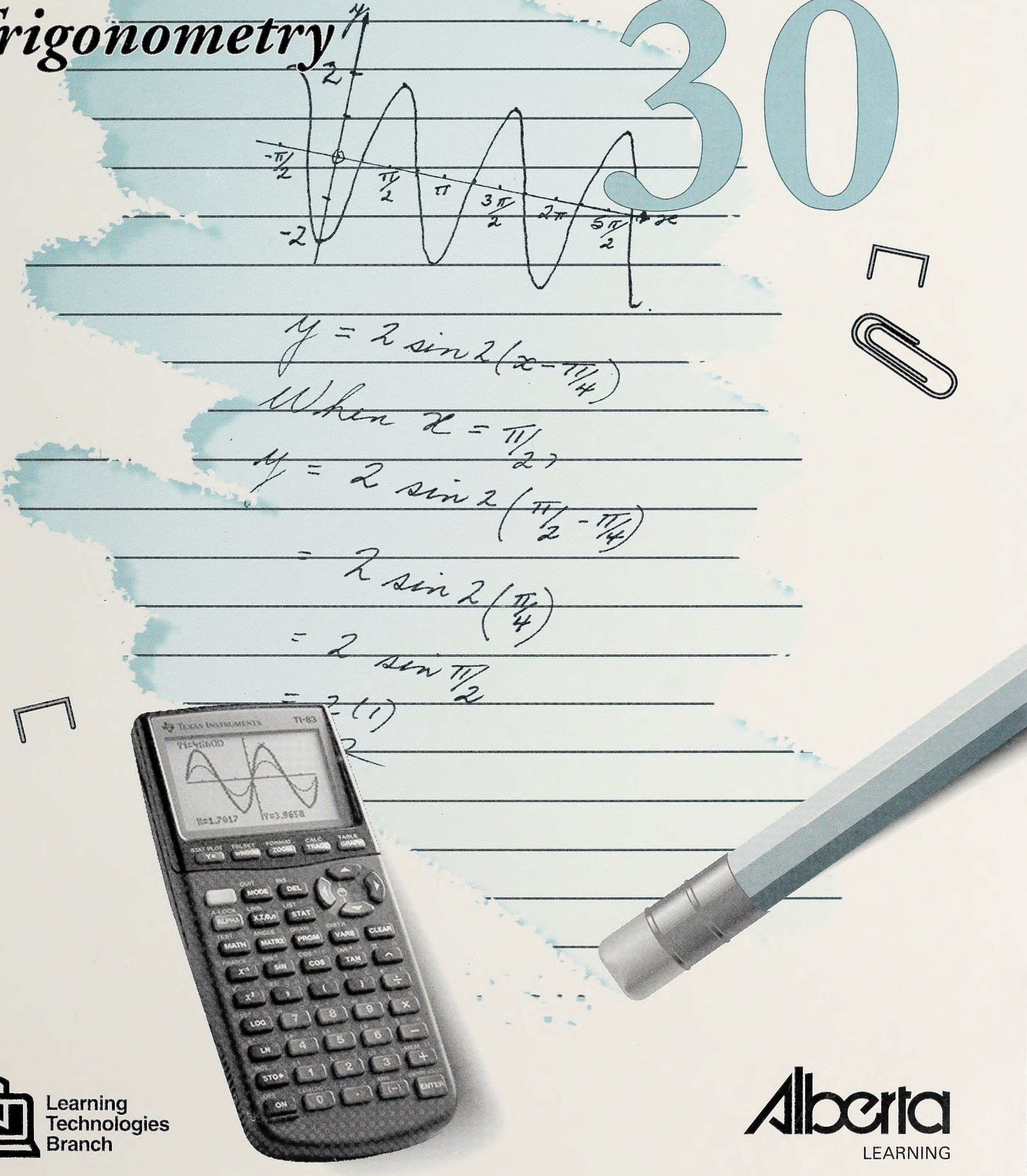
Module 5

MATHEMATICS 30

Trigonometry



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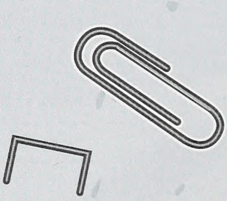
Pure MATHEMATICS 30

Module 5

Trigonometry



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Pure Mathematics 30
Module 5: Trigonometry
Student Module Booklet
Learning Technologies Branch
ISBN 0-7741-1769-9

This document is intended for	
Students	✓
Teachers	✓
Administrators	
Parents	
General Public	
Other	

The Learning Technologies Branch has an Internet site that you may find useful. The address is as follows:

http://www.learning.gov.ab.ca/ltb

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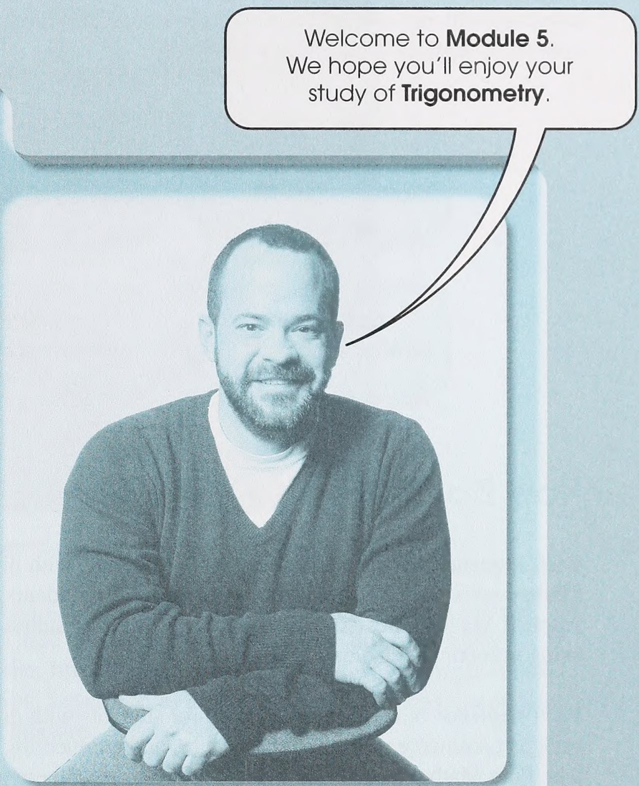
Welcome

Pure Mathematics 30

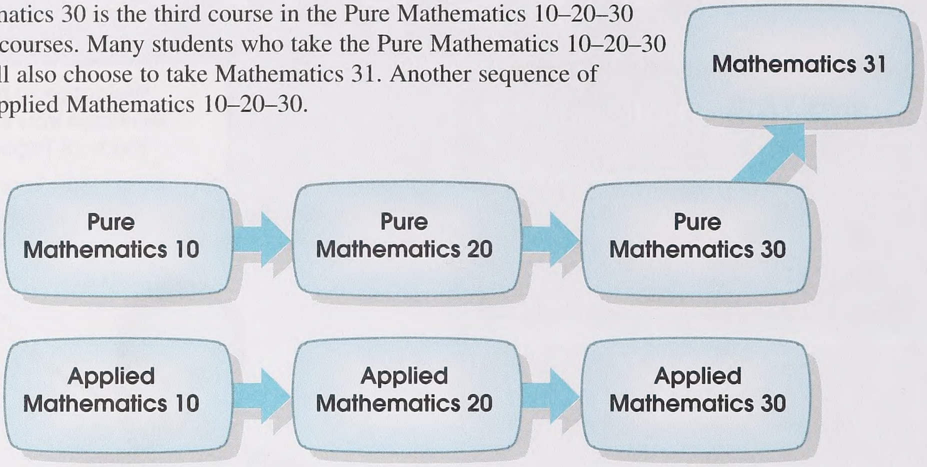
- **Module 1** Transformations
- **Module 2** Exponents and Logarithms
- **Module 3** Sequences and Series
- **Module 4** Conics
- **Module 5** Trigonometry
- **Module 6** Combinatorics
- **Module 7** Probability and Statistics


Welcome to **Module 5**. We hope you'll enjoy your study of **Trigonometry**.

Pure Mathematics 30 contains seven modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.

High School Mathematics Courses

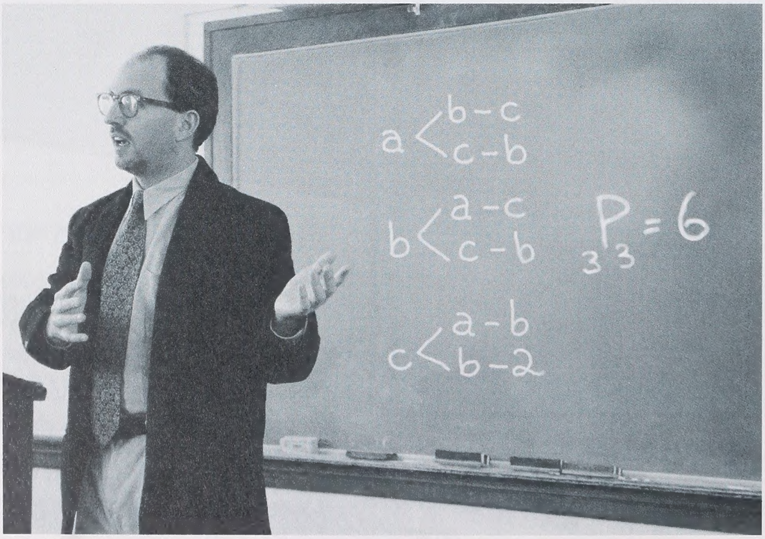
Pure Mathematics 30 is the third course in the Pure Mathematics 10–20–30 sequence of courses. Many students who take the Pure Mathematics 10–20–30 sequence will also choose to take Mathematics 31. Another sequence of courses is Applied Mathematics 10–20–30.



How Do the Sequences Differ?

Each sequence of courses is designed for students with different mathematical strengths and interests. The Pure Mathematics 10–20–30 sequence is intended for students who are strong in mathematical theory and algebra. The Applied Mathematics 10–20–30 sequence is better suited to students who prefer to solve mathematical problems using numerical reasoning or geometry.

Each sequence of courses is designed for students with different career plans. Students who plan to take courses such as commerce, engineering, physics, and engineering technologies in post-secondary education will need to take Pure Mathematics 30 in high school. Some of these students will also need to take Mathematics 31.



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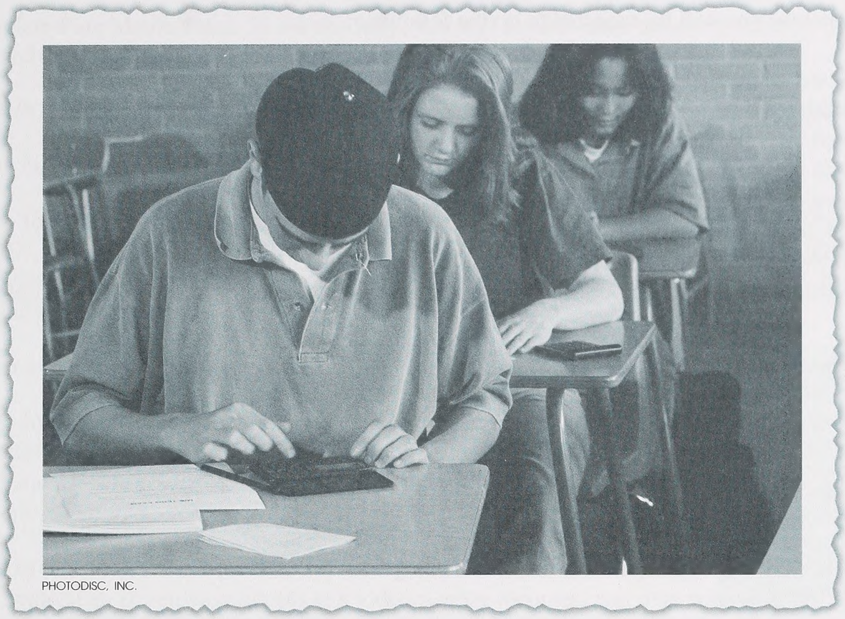
Where Can I Obtain Diploma Examination Information?



PHOTODISC, INC.

Alberta Students will write a diploma examination at the end of the course. Alberta Learning provides several documents to help students prepare for this examination. These documents are found in the directory under "Kindergarten to Grade 12" on the Alberta Learning website, **http://www.learning.gov.ab.ca**. Information like course expectations, the makeup of the diploma examination, keyed copies of previous examinations, preparation guides, and calculator policies are available to students at this site.

Each year, in February and September, Alberta Learning provides teachers with information on a **student project**, which teachers **may** use as part of your overall assessment. Information to students will also be posted on the Alberta Learning website. Check with your teacher to determine what you will be expected to do. Be aware that one of the diploma examination's written-response questions is worth 10% of your diploma examination mark and will deal with elements of this project.

You should take advantage of the many sources of information about Pure Mathematics 30. Your success depends on your understanding of course expectations and evaluation procedures. Work closely with your teacher and do not hesitate to ask questions.

Remember, take the initiative to find out all you can about Pure Mathematics 30.

Can I Transfer to the Applied Mathematics Sequence of Courses?

If you do not successfully complete Pure Mathematics 30, you can repeat the course or you can transfer to the Applied Mathematics sequence. If you decide to transfer to the Applied Mathematics sequence, you may take the five-credit course called Applied Mathematics 20b or you may take the five-credit course called Applied Mathematics 20.

Mathematical Processes

Throughout the course you will be expected to do the following:

- communicate mathematically
- connect mathematical ideas to everyday experiences and to concepts in other disciplines
- develop and use estimation strategies
- develop and use mental math strategies
- develop and use problem-solving strategies
- reason and justify your thinking
- select and use appropriate technologies as tools to solve problems
- use visualization to assist in processing information, making connections, and solving problems

Mathematical Process

■ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

This icon will show which mathematical process you are expected to perform.

In order to develop these mathematical processes more fully, you are encouraged to ask someone who is also taking Pure Mathematics 30 to be your study partner. You may find that having a friend to discuss mathematics with makes your studying more enjoyable.

You are also encouraged to record your mathematical ideas in a journal. Here are some suggestions for organizing your journal:

- Keep a section on new concepts and procedures you have learned. Get in the habit of describing new concepts, procedures, and strategies in your own words. If you are having difficulty, write your questions in your journal before discussing them with your teacher or study partner. Record useful ways to help you remember what a concept means. Make graphic organizers (such as context webs, Venn diagrams, or organizational charts) to help you connect mathematical ideas.

- Keep a section on your mathematical accomplishments. This can include solutions to problems that you are proud of solving. It can also include landmark events, such as when you finally grasped a difficult concept (an “aha!”) or when you first used a calculator or spreadsheet program in a new way.

- Keep a section on mathematics in the everyday world. This could include newspaper articles, cartoons, and descriptions of careers that require a mathematical background.

Resources You Will Need

In order to complete Pure Mathematics 30 successfully, you will need to have access to the following resources, which can be purchased through the Learning Resources Distributing Centre:

- the *MATHPOWER™ 12* textbook, Western Edition, published by McGraw-Hill Ryerson (2000)
- a graphing calculator

  Any of the following graphing calculators may be used: *Texas Instruments* TI-83, TI-83 Plus, TI-82, TI-86, TI-89, TI-92, or TI-92 Plus; *Casio* CFX-9850Ga-Plus, CFX-9850G, CFX-9800G, or FX-9700 series; or *Sharp* EL-9600c, EL-9600, EL-9300, or EL-9200. **Note:** Some of these calculators are no longer commercially manufactured.

- a computer and a spreadsheet program

  Any of the following spreadsheet programs may be used: *ClarisWorks™ 4.0* (or higher) for Macintosh, *ClarisWorks™ 4.0* (or higher) for Windows, *Microsoft® Works 3.0* (or higher) for Windows, or *Microsoft® Access 97* and *Microsoft® Excel 97* for Windows (a part of *Microsoft® Office Professional 95* or *Microsoft® Office Professional 97*).

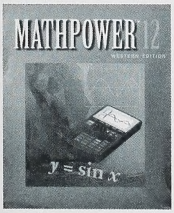
Note: Examples in this course show the TI-83 graphing calculator and the *Microsoft® Excel 97* spreadsheet program where applicable. If you have access to a videocassette recorder, you may wish to view the video *The TI-83 Graphing Calculator Video Tutor* to review some of the calculator's features. Check with your distance learning provider concerning the availability of this video.

Finally, you will need the following resources, which can be purchased locally:

- a mathematical instrument set (compass, protractor, ruler, and triangles)
- a notebook or binder in which to respond to the questions asked
- a notebook or binder for journal writing

Other Visual Cues

In addition to the Mathematical Process icon described earlier, you will find other visual cues throughout this course. Read the explanations given to discover what each icon prompts you to do.



Refer to the textbook.

View a videocassette.

Use the Internet to explore a topic.

Answer the questions in the Assignment Booklet.

Use the Pure Mathematics 30 Companion CD.

Remember: *Any Internet website address given in this module is subject to change.*

CONTENTS

Module Overview

Evaluation
2
Strategies for Completing a Module
3

Section 1

Trigonometric Functions

Activity 1: Angular Measure
6
Activity 2: Trigonometric Ratios
15
Activity 3: Graphing Sine and Cosine Functions
23
Activity 4: Applications of Sinusoidal Functions
34
Activity 5: Other Trigonometric Functions
40
Follow-up Activities
44
Extra Help
44
Enrichment
51
Conclusion
55
Assignment
55

Section 2

Trigonometric Equations

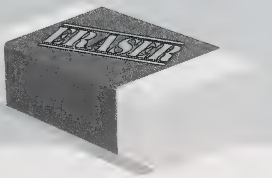
Activity 1: Relating Graphs and Solutions
58
Activity 2: Solving Trigonometric Equations
64
Activity 3: Using Technology
70
Activity 4: Trigonometric Identities
76
Activity 5: Sum, Difference, and Double-Angle Identities
82
Follow-up Activities
85
Extra Help
85
Enrichment
89
Conclusion
91
Assignment
91

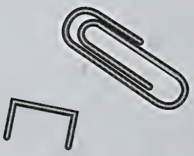
Module Summary

Final Module Assignment
92

Appendix

Glossary
94
Suggested Answers
94
Credits
264





Module Overview

Even though the term *trigonometry* is derived from the Greek words for triangle measurement, this branch of mathematics has many applications beyond the triangle. Events that occur in a cyclical or periodic manner can be modelled using trigonometric functions. The phases of the moon, sound and light waves, a swinging pendulum, the rise and fall of the tides, the location of Earth in its orbit, the change in average temperatures over the course of a year, and sunset times are just a few examples. Trigonometry is essential to engineering, the sciences, and most aspects of mathematics.

In this module, you will examine the periodic properties of trigonometric functions. In Section 1, you will explore angular measure and the definitions and graphs of the trigonometric functions; then you will use trigonometric functions to model periodic events and solve related problems.

In Section 2, you will solve equations involving trigonometric functions using pencil-and-paper techniques and technology. You will then explore a special class of equations, called identities. Throughout your study, you will apply your newly acquired skills to practical problem-solving situations.

Module 5: Trigonometry
Section 1: Trigonometric Functions
Section 2: Trigonometric Equations

Evaluation

The document you are presently reading is called a Student Module Booklet. It will show you, step by step, what to do and how to do it.

This module, Trigonometry, has two sections. Within each section, your work is grouped into activities. Within the activities, there are readings and questions for you to answer. By completing these questions you will construct your own learning, discover mathematical connections, and apply what you have learned. The suggested answers in the Appendix of this Student Module Booklet will provide you with immediate feedback on your progress.

In this module, you will be directed to the accompanying Assignment Booklets. You are expected to complete two Assignment Booklets for this module. Your grading in this module is based upon the assignments that you submit for evaluation. The mark distribution is as follows:

Assignment Booklet 5A	
Section 1 Assignment	**30 marks**
Assignment Booklet 5B	
Section 2 Assignment	**35 marks**
Final Module Assignment	**35 marks**
TOTAL	**100 marks**

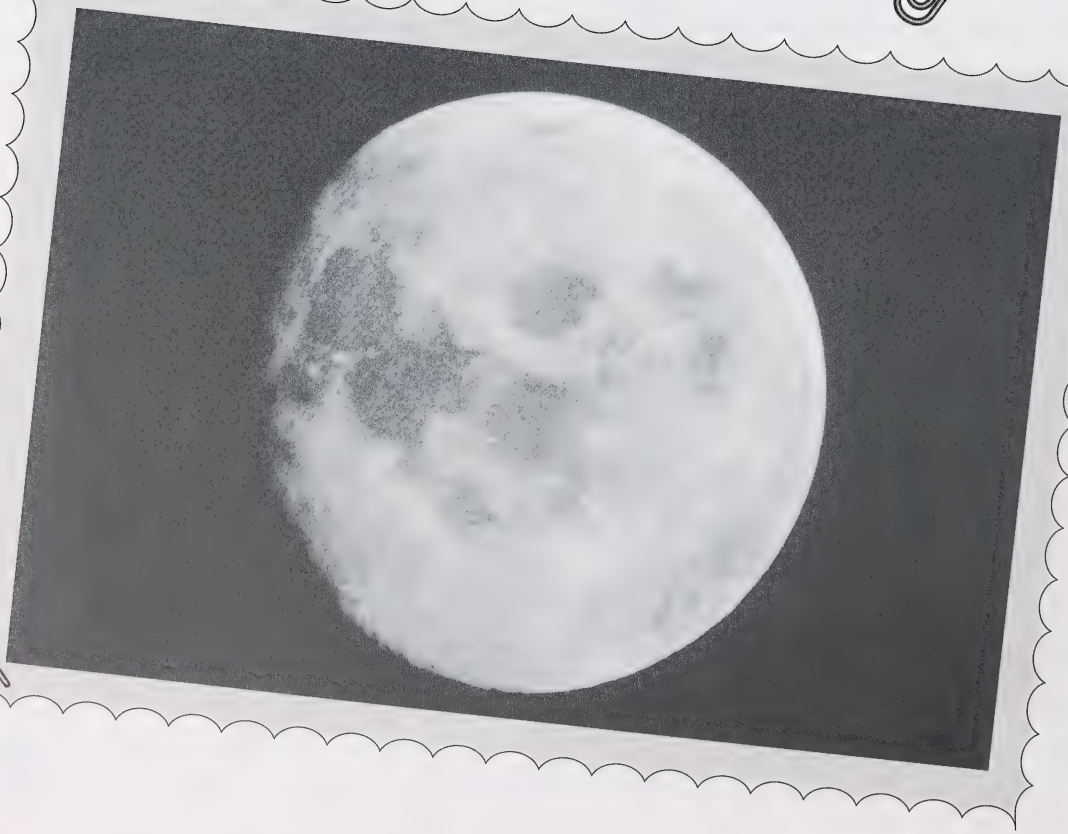
Strategies for Completing a Module

Before you begin, organize your materials: Student Module Booklet, textbook, Assignment Booklets, binder, lined paper, graph paper, graphing calculator, pens, pencils, and so on. Make sure you have a quiet area in which to work, away from distractions. Set up your time schedule.

To achieve success in this module, be sure to read all your instructions carefully and work slowly and systematically through the material. Remember, it's the work you do in this Student Module Booklet that will prepare you for your assignments and final test. Try to set realistic goals for yourself each day; and once you've set them, stick to them. Submit your assignments regularly, and don't forget to review your work before handing it in. Careful work habits will greatly increase your chances for success in Pure Mathematics 30.

Good luck!

Section 1



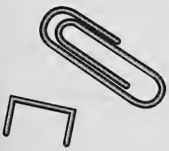
Trigonometric Functions

Throughout history, the full moon has kindled the imagination. The harvest moon—the full moon closest to the first day of autumn—fills the evening sky, bathing the landscape in brilliant moonlight and marking the passage of summer. The next full moon, or hunter's moon, does not appear as early in the evening. The phases of the moon recur at regular intervals, about every month or 29 1/2 days.

Did you know that Earth and the Moon form a system? Did you know that every 27 1/3 days, they both revolve about the system's centre of mass, located a little more than 4500 km from the centre of Earth? It is this centre of mass that traces an elliptical path around the Sun. Both Earth and the Moon crisscross this orbit on their respective journeys.

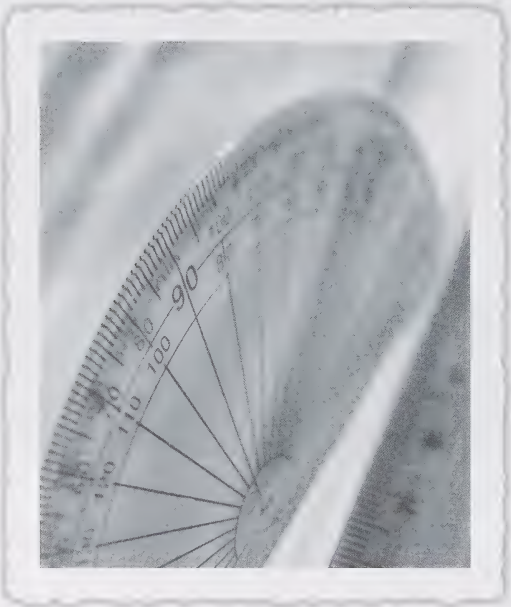
Periodic motion—motion that recurs at regular intervals like the motion of the Moon or the motion of Earth—can be modelled using trigonometric functions.

In this section, you will examine angular measure and the definitions of the trigonometric functions. You will explore the graphs of the six trigonometric functions, and you will apply the graphs of the sine and cosine functions to model periodic events.



Activity 1: Angular Measure

You can thank the people of ancient Mesopotamia (the Sumerians and Babylonians) for dividing the circle into 360 degrees. Their sexagesimal number system (a number system based on 60 and still used in angle measurement) may have arisen because there are approximately 360 days in a year and 360 is a multiple of 60. A second and purely practical reason they adopted 60 as the base is because 60 is divisible by 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60—facilitating the handling of fractional amounts. Besides angle measure, time also uses a sexagesimal system, with 60 seconds in a minute and 60 minutes in an hour.



Mathematical Process

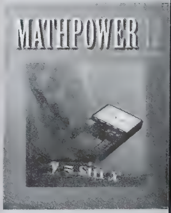
☐ Communication
☒ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

a unit of angular measure determined by the size of a circle's central angle subtended by an arc the same length as the radius

If you have access to the Internet, you can discover more about Mesopotamian mathematics and the sexagesimal system at the following website:

http://it.stlawu.edu/~dmelvill/mesomath/sumerian.html

A second system of angle measure is based on radian measure. A radian is the measure of a circle's central angle that is subtended by an arc equal to the circle's radius in length. In this activity, you will explore radian measure and its relationship to degree measure.



Turn to page 186 of *MATHPOWER™ 12* and read "Angular Measure."

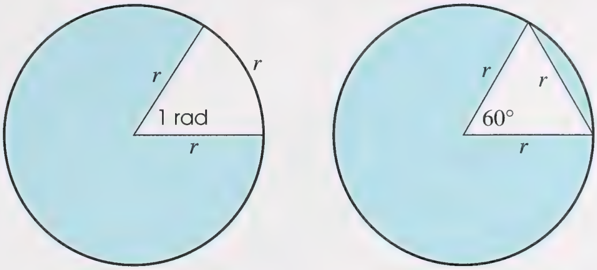
1. Answer the following questions on page 186 of the textbook:

   a. questions a. and b. of "Explore: Use a Model to Visualize"
   b. questions 1 to 3 of "Inquire"

Turn to page 94 to compare your responses with those in the Appendix.

Mathematical Process
- ■ Communication
- ☒ Connection
- ☒ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

The preceding questions introduced you to radian measure. Remember, 1 radian, abbreviated 1 rad (as shown in the circle on the left), is the measure of the central angle in a circle subtended by an arc equal to the radius in length.



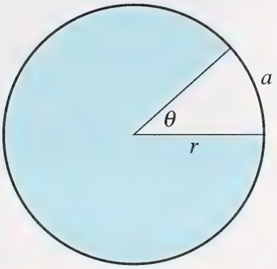
One radian is slightly smaller than the 60° angle drawn in the triangle on the right. The 60° angle is subtended by a chord equal to the radius in length. For the radian shown in the circle on the left, the curvature of the circle reduces the central angle.

Because the circumference of a circle is $2\pi r$, there are 2π radians in a circle.

$$\begin{aligned}\therefore\ 2\pi \text{ rad} &= 360^\circ\\ 1 \text{ rad} &= \frac{360^\circ}{2\pi}\\ &\doteq 57.3^\circ\end{aligned}$$

To examine radian measure further, open the Computer Assisted Instruction folder on the Pure Mathematics 30 Companion CD and work through the segment titled *Radian Measure.*

The relationship among the measure (in radians) of a central angle, θ, of a circle, the arc, a, that subtends this angle, and the radius, r, of the circle is as follows:



$$\theta = \frac{a}{r} \qquad \text{or} \qquad a = r\theta$$

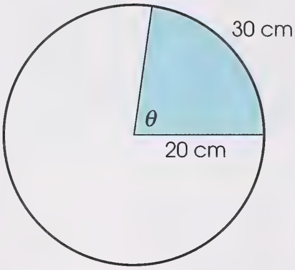
Because θ is a ratio of two lengths, it is a pure number; thus, 3 rad can be written simply as 3. This value means the central angle is subtended by an arc 3 times longer than the radius, regardless of the units used to measure the arc and radius.

Example

Determine the measure of the central angle subtended by a 30-cm arc along the circumference of a circle with a diameter of 40 cm.

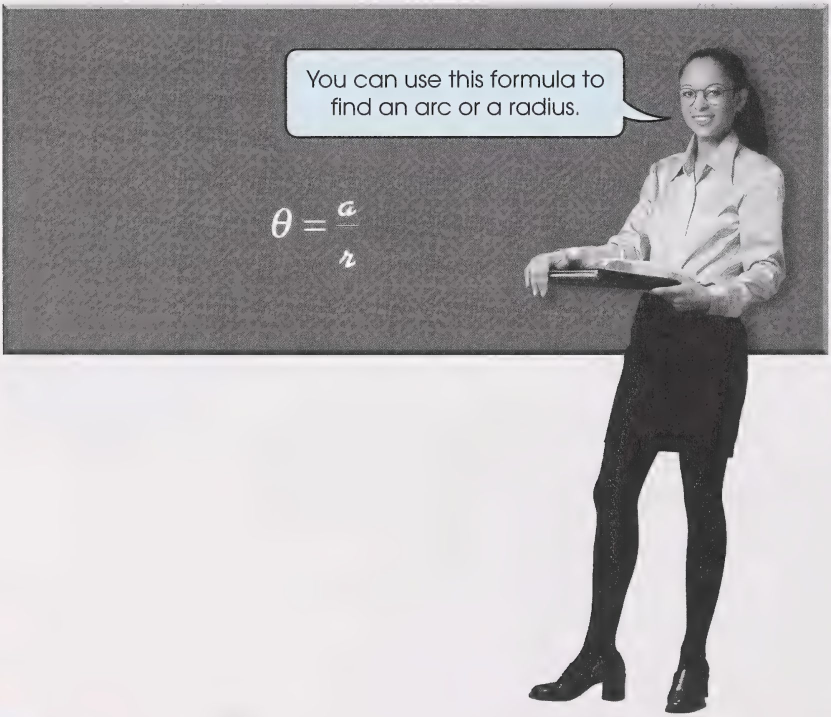
Solution

$$
\begin{aligned}
d &= 2r \\
40 &= 2r \\
r &= 20
\end{aligned}
$$



$$
\begin{aligned}
\therefore\ \theta &= \frac{a}{r} \\
&= \frac{30}{20} \\
&= 1.5
\end{aligned}
$$

The central angle is 1.5 rad (or simply 1.5).


You can use this formula to find an arc or a radius.
$\theta = \frac{a}{r}$

Example

A wheel of a bicycle passes through a puddle and leaves a wet mark 40-cm long on the pavement. If the wheel rotated through 1.12 rad, what is the radius of the wheel? Round your answer to the nearest centimetre.

Solution

$a = 40$ and $\theta = 1.12$

$$\begin{aligned} \therefore\ \theta &= \frac{a}{r} \\ 1.12 &= \frac{40}{r} \\ r &= \frac{40}{1.12} \\ &\doteq 36 \end{aligned}$$

The radius of the wheel is approximately 36 cm.

Radian measure greatly simplifies calculations for people like physicists and mathematicians. Mathematicians prefer radians for other, less utilitarian reasons as well, which include the simple and elegant way radians describe the measure of an angle.

Mathematical Process

- ■ Communication
- ☒ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

Because many of the angle measurements you see are given in degree form, it is useful to know a way of converting between the radian system and the degree system. There is a straightforward way to change radian measure to degree measure and vice versa. Recall that the circumference of a circle is 2π when the radius is 1; thus, a whole circle has a measure of 2π rad. Also, recall that a whole circle is 360°.

$$\begin{aligned} \therefore\ 2\pi \text{ rad} &= 360^\circ \\ \pi \text{ rad} &= 180^\circ \end{aligned}$$

Now, when you want to convert from radians to degrees, you can use the following relationship:

$$\begin{aligned} \pi \text{ rad} &= 180^\circ \\ \frac{\pi \text{ rad}}{\pi} &= \frac{180^\circ}{\pi} \\ 1 \text{ rad} &= \frac{180^\circ}{\pi} \end{aligned}$$

Also, when you want to convert from degrees to radians, you can use the following relationship:

$$\pi \text{ rad} = 180°$$

$$\frac{\pi \text{ rad}}{180} = \frac{180°}{180}$$

$$\frac{\pi}{180} \text{ rad} = 1°$$

$$1° = \frac{\pi}{180} \text{ rad}$$

Example

a. An angle measures $\frac{\pi}{3}$ rad. What is the measure of this angle in degrees?

b. An angle measures 135°. What is the measure of this angle in radians?

Solution

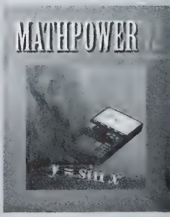
a.

$$\begin{aligned} 1 \text{ rad} &= \frac{180°}{\pi} \\ \frac{\pi}{3} \text{ rad} &= \frac{\cancel{\pi}}{3}\left(\frac{180°}{\cancel{\pi}}\right) \\ &= \frac{180°}{3} \\ &= 60° \end{aligned}$$

b.

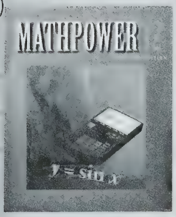
$$\begin{aligned} 1° &= \frac{\pi}{180} \text{ rad} \\ 135° &= \overset{3}{\cancel{135}}\left(\frac{\pi}{\underset{4}{\cancel{180}}} \text{ rad}\right) \\ &= \frac{3\pi}{4} \text{ rad or } \frac{3}{4}\pi \text{ rad} \end{aligned}$$

Turn to page 187 of *MATHPOWER™ 12* and work through Examples 1 and 2.



2. Answer questions 1, 2, 4, 5, 6, 10, 11, 12, 13, 16, 17, and 20 of "Practice," on pages 189 and 190 of the textbook.

Turn to page 95 to compare your responses with those in the Appendix.



Mathematical Process

☐ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☒ Technology
☐ Visualization

Turn to "TECHNOLOGY" on page 192 of *MATHPOWER™ 12* and read "Investigating Snell's Law Using a Spreadsheet."

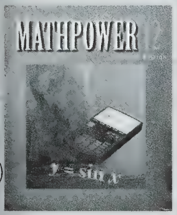
3. Answer the following questions on pages 192 and 193 of the textbook:

a. questions 1 and 2 of Investigation 1, "Finding the Index of Refraction of Water"

b. questions 1 and 2 of Investigation 2, "Determining the Index of Refraction of Other Substances"

Turn to page 96 to compare your responses with those in the Appendix.

Arc Lengths and Sector Angles



Mathematical Process

☐ Communication
☐ Connection
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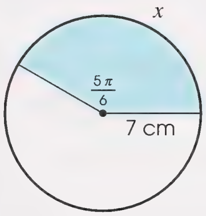
☒ Problem Solving
☒ Reasoning
☐ Technology
☐ Visualization

Turn to page 188 of *MATHPOWER™ 12* and read from the top of the page to the red line, working through Example 3.

Example

Find the measure of the arc shown. Round your answer to the nearest tenth.



Solution

$$\frac{\text{arc length}}{2\pi r} = \frac{\text{sector angle}}{2\pi}$$

$$\frac{x}{2\pi(7)} = \frac{\frac{5\pi}{6}}{2\pi}$$

$$\frac{x}{7} = \frac{5\pi}{6}$$

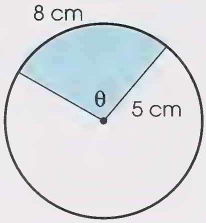
$$x = \frac{35\pi}{6}$$

$$\doteq 18.3$$

The measure of the arc is about 18.3 cm.

Example

Determine the sector angle in radians. Round your answer to the nearest tenth.



Solution

$$\frac{\text{arc length}}{2\pi r} = \frac{\text{sector angle}}{2\pi}$$

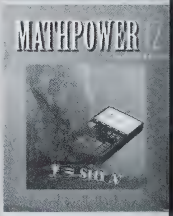
$$\frac{8}{2\pi(5)} = \frac{\theta}{2\pi}$$

$$\frac{8}{5} = \theta$$

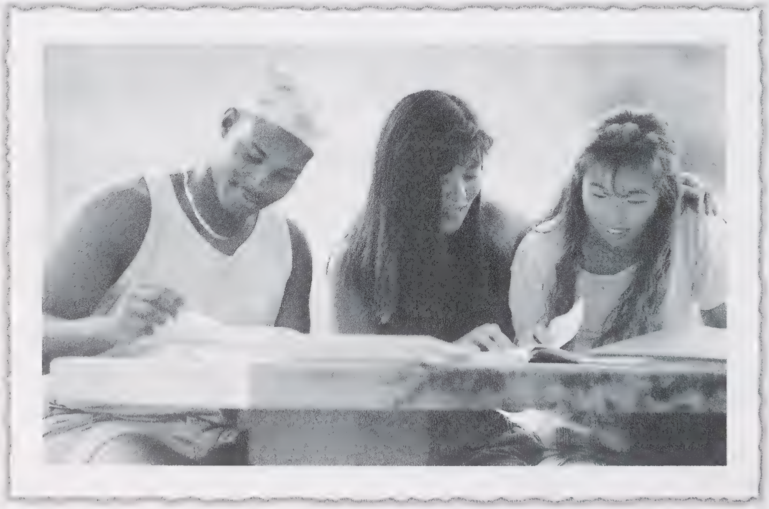
$$\theta = 1.6$$

The measure of the sector angle is 1.6 rad.

4. Turn to page 190 of *MATHPOWER™ 12* and answer questions 21, 24, 26, and 27 of "Practice."



Turn to page 98 to compare your responses with those in the Appendix.

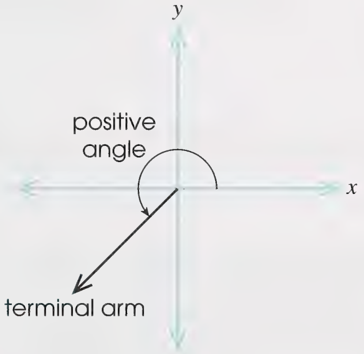


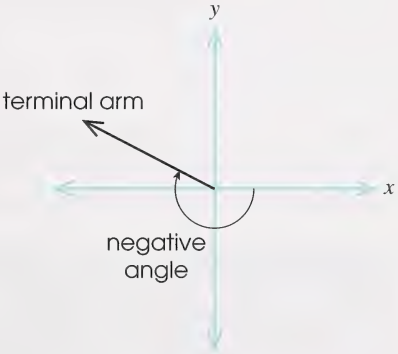
Coterminal Angles

standard position: the location of an angle in the plane in which the vertex is at the origin, the initial arm lies along the positive x-axis, and the terminal arm is free to rotate

In Pure Mathematics 10, you defined trigonometric ratios in terms of the lengths of the sides of a right triangle. In this course, you will define the trigonometric ratios in terms of angles drawn in standard position.

The following angles are in standard position with their initial rays along the positive *x*-axis. Notice that the measure of an angle is considered positive when the rotation from the initial ray to the terminal ray is counterclockwise. The measure of an angle is considered negative when the rotation is clockwise.



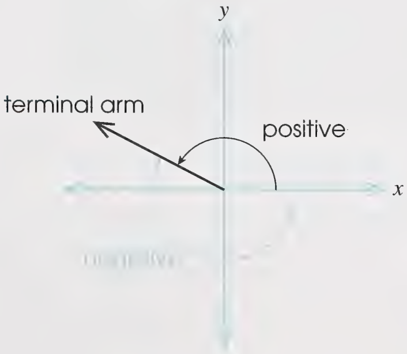


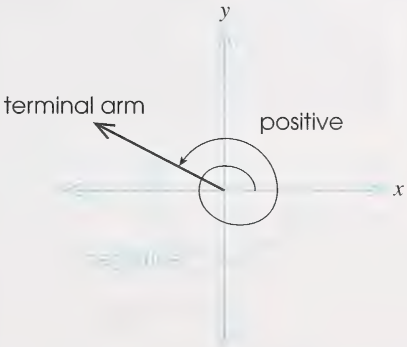
From the initial and terminal arms alone, you cannot determine the measure of an angle or even if it is positive or negative. In fact, there is an infinite number of possibilities.

[illegible]: angles in standard position with the same initial arm and terminal arm

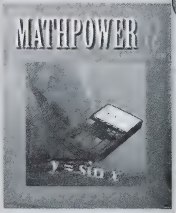
Different angles with the same terminal and initial arms are called coterminal angles.

The following diagram illustrates how you could represent a given angle three different ways.





Turn to page 188 of *MATHPOWER™ 12* and read from the red line to the red line near the bottom of page 189, working through Examples 4 and 5.



5. Answer the following questions on pages 190 and 191 of the textbook:

- **a.** questions 29, 31, 35, 39, 41, 42, 45, 50, 51, 54, 59, 64, 68, and 70 of "Practice"
- **b.** questions 71, 72, 74, 75, and 77 of "Applications and Problem Solving"

Turn to page 98 to compare your responses with those in the Appendix.

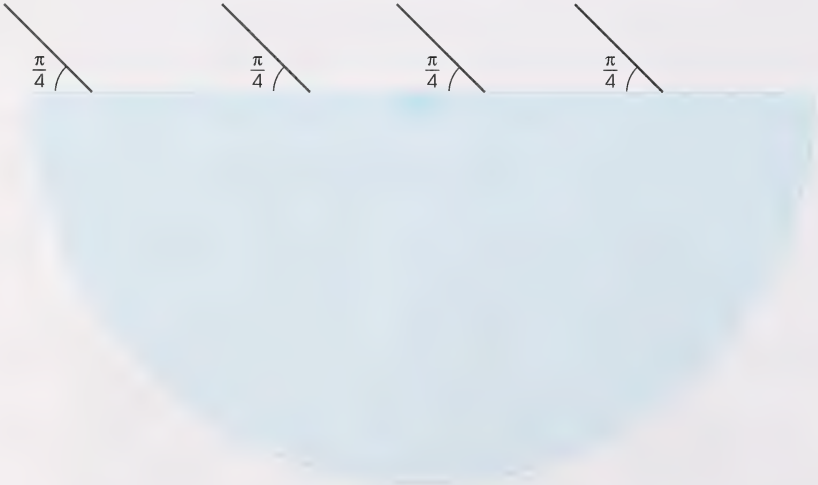
Now Try This

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ☒ Problem Solving
- ☒ Reasoning
- ■ Technology
- ■ Visualization

Earlier in this activity, you worked with Snell's formula for determining the index of refraction of objects. Particularly, you calculated the index of refraction of glass and drew a diagram showing the passage of a ray of light through a plate of glass.

6. Four rays of light strike a glass hemisphere (as shown in the diagram). Each ray makes an angle of $\frac{\pi}{4}$ with the surface. Assuming the index of refraction is 1.56, where do these rays exit from the hemisphere and in what direction are they headed?



Turn to page 103 to compare your response with the one in the Appendix.

Looking Back

Mathematical Process

☒ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

In this activity, you examined a second measuring system for angles, called radian measure. You converted from the degree system to the radian system and vice versa.

Bertrand Russell, a famous mathematician and philosopher, wrote, “Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.”

In your journal, explain how Russell’s quote would apply to the radian system for measuring angles.

Activity 2: Trigonometric Ratios

Mathematical Process

☐ Communication
☒ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

In modern society, surveying is a very important part of land ownership. People don’t want to have to tear down a structure because part of it was built on the neighbour’s land. When new developments are being built, whether they are housing developments or highways and roads, you see surveyors and their transits out making sure that everything is in the right place.

Have you ever wondered how a transit works and what mathematics is needed to make use of the measurements taken by a surveyor?

If you have access to the Internet, you can discover a great deal about the history of surveying and its modern applications from the following website:

http://www.surveyhistory.org

You are about to study trigonometric functions. These functions appear in calculations surveyors and engineers perform; they also show up in other surprising places. In this activity, you will examine the definitions of the primary and reciprocal trigonometric ratios and use the definitions to calculate exact values for many special angles.

Turn to page 194 of *MATHPOWER™ 12* and read "Trigonometric Ratios of Any Angle."

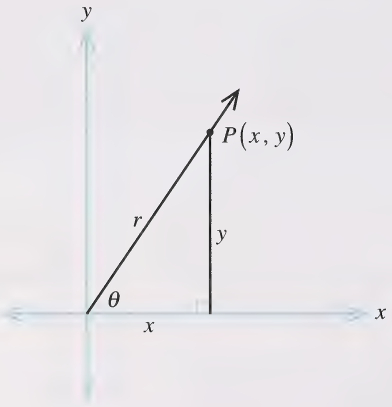
1. Answer the following questions on page 194 of the textbook:

- **a.** questions a. and b. of "Explore: Use a diagram"
- **b.** questions 1 to 7 of "Inquire"

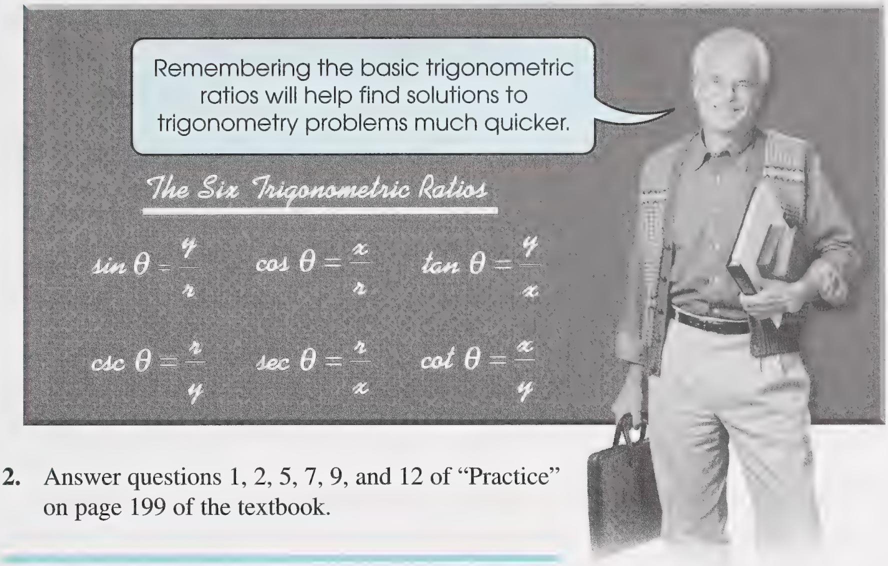
Turn to page 104 to compare your responses with those in the Appendix.

Mathematicians often gain insight into general solutions to problems by looking at specific examples. Questions 6 and 7 of "Inquire," on page 194 of the textbook, are specific examples that can be generalized into a definition of the trigonometric ratios for any angle. Point P is chosen as an arbitrary point on the terminal ray of an arbitrary angle, θ, in standard position.

The lengths r, x, and y are related by the Pythagorean Theorem, $r=\sqrt{x^2+y^2}$. Keep in mind that the values of x and y can be positive or negative depending on the location of point P in the plane; however, the value of r is always positive!



Turn to page 195of *MATHPOWER™ 12* and read from the top of the page to the red line, working through Example 1.


Remembering the basic trigonometric ratios will help find solutions to trigonometry problems much quicker.
The Six Trigonometric Ratios
$\sin\theta = \frac{y}{r}$
$\cos\theta = \frac{x}{r}$
$\tan\theta = \frac{y}{x}$
$\csc\theta = \frac{r}{y}$
$\sec\theta = \frac{r}{x}$
$\cot\theta = \frac{x}{y}$

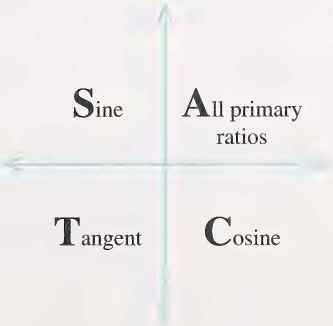
2. Answer questions 1, 2, 5, 7, 9, and 12 of “Practice” on page 199 of the textbook.

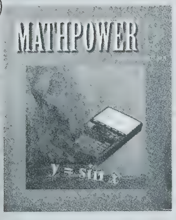
Turn to page 106 to compare your responses with those in the Appendix.

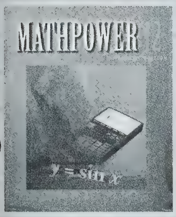
In the preceding questions, did you notice that all of the ratios were positive when the angles appeared in the first quadrant? For angles appearing in the other three quadrants, some of the trigonometric ratios were negative and some were positive.

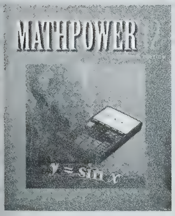
Turn to page 195 of *MATHPOWER™ 12* and read from the red line to the red line on page 196.

The summary chart shows that all of the ratios are positive in Quadrant I. In each of the other quadrants, only one of the primary trigonometric ratios is positive. In Quadrant II, the sine ratio is positive; in Quadrant III, the tangent ratio is positive; and in Quadrant IV, the cosine ratio is positive. This is briefly summarized in the illustration on the right.





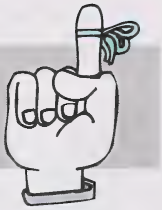




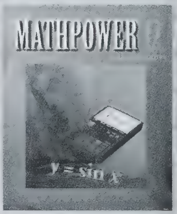
Mathematical Process

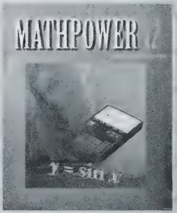
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☒ Visualization

The letters of the positive ratios spell CAST, a useful mnemonic to help you remember which ratios are positive in each quadrant.



3. Answer questions 13 to 20 of "Practice" on page 199 of the textbook.





Turn to page 109 to compare your responses with those in the Appendix.

Turn to page 196 of *MATHPOWER*™ *12* and read from the red line to the bottom of the page. **Note:** In the table, 45° should be represented by $\frac{\pi}{4}$ rad. Make this correction in your textbook.

The special angles—0°, 30°, 45°, 60°, and 90°—have trigonometric ratios that are easily expressed in exact form. It might seem that a value correct to 10 or 12 decimal places is good enough; but that is not always the case. For example, if you are trying to verify an identity, a small difference might mislead you. (You will study more about identities in Section 2: Activity 3.) Also, an exact value like $\frac{\sqrt{2}}{2}$ might suggest a solution that 0.707 106 7 might not.

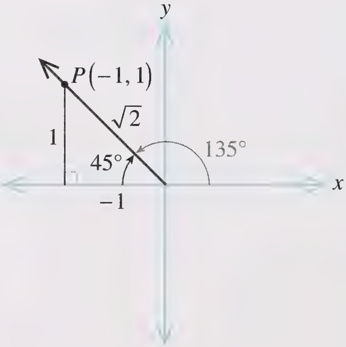
Most of the special angles come from a right angle being divided by 2 or 3, which makes them easy to construct and visualize. The simple form of sine and cosine for these angles makes them particularly easy to remember and use.

Example

Find the exact primary trigonometric ratios for 135°.

Solution

The diagram shows that a 45° angle is formed between the terminal arm and the negative x-axis. This 45° angle is known as the reference angle for the 135° angle. Build a 45°-45°-90° triangle along the negative x-axis. Thus, the coordinates of point P on the terminal arm of 135° are $(-1, 1)$.



reference angle: the acute angle formed by the terminal arm of an angle in standard position and the x-axis

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

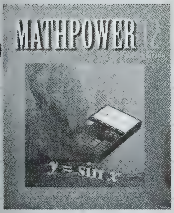
$$\sin 135^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cos 135^\circ = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad \tan 135^\circ = \frac{1}{-1} = -1$$

Notice that the acute angle, 45°, and the given angle, 135°, have the same trigonometric ratios except for the signs.

$$\sin 45^\circ = \frac{\sqrt{2}}{2} \qquad \sin 135^\circ = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{\sqrt{2}}{2} \qquad \cos 135^\circ = -\frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = 1 \qquad \tan 135^\circ = -1$$



Turn to page 197 of *MATHPOWER™ 12* and work through Example 2.

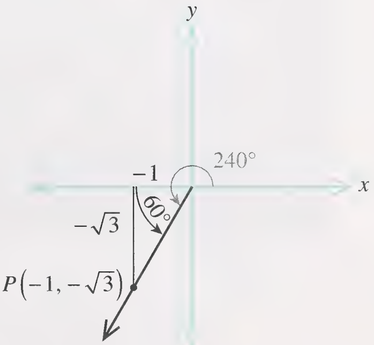
Reference angles are useful in finding the trigonometric ratios for angles that do not fall in the first quadrant. If you know the values for the trigonometric ratios of the reference angle, you can figure out the values for the given angle by adjusting the signs.

Example

Find the exact primary trigonometric ratios for 240°.

Solution

The diagram shows that a 60° angle is formed between the terminal arm and the negative x-axis. This 60° angle is the reference angle for the 240° angle. Build a 30°-60°-90° triangle along the negative x-axis. Thus, the coordinates of point P on the terminal arm of 240° are $\left(-1, -\sqrt{3}\right)$.



$$\begin{aligned} \sin\theta &= \frac{y}{r} \\ \sin 240^\circ &= \frac{-\sqrt{3}}{2} \\ &= -\frac{\sqrt{3}}{2} \end{aligned} \qquad \begin{aligned} \cos\theta &= \frac{x}{r} \\ \cos 240^\circ &= \frac{-1}{2} \\ &= -\frac{1}{2} \end{aligned} \qquad \begin{aligned} \tan\theta &= \frac{y}{x} \\ \tan 240^\circ &= \frac{-\sqrt{3}}{-1} \\ &= \sqrt{3} \end{aligned}$$

You could have used the reference angle 60° and the CAST rule to determine the primary trigonometric ratios.

$$\sin 60^\circ = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{1}{2} \qquad \tan 60^\circ = \sqrt{3}$$

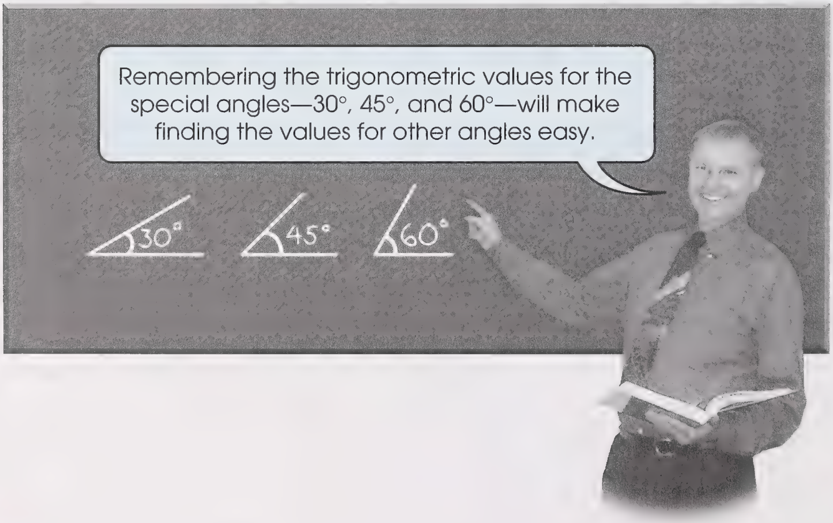
Because only tangent is positive in the third quadrant, $\sin 240^\circ = -\frac{\sqrt{3}}{2}$, $\cos 240^\circ = -\frac{1}{2}$, and $\tan 240^\circ = \sqrt{3}$.

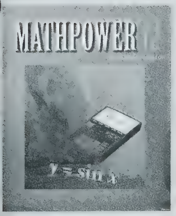
4. Turn to page 200 of *MATHPOWER™ 12* and answer questions 22, 24, and 26 of "Practice."

Turn to page 111 to compare your responses with those in the Appendix.

Mathematical Process

☐ Communication
☐ Connection
☐ Estimation
☒ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization


Remembering the trigonometric values for the special angles—30°, 45°, and 60°—will make finding the values for other angles easy.
30°
45°
60°

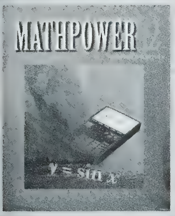


Turn to pages 197 and 198 of *MATHPOWER™ 12* and work through Examples 3 and 4.

5. Answer the following questions on page 200 of the textbook:

- **a.** questions 27, 28, 31, and 33 of "Practice"
- **b.** questions 43, 44, 45, and 47 of "Applications and Problem Solving"

Turn to page 113 to compare your responses with those in the Appendix.



Turn to page 199 of *MATHPOWER™ 12* and work through Example 5. Pay particular attention to the importance of a suitable diagram in the solution of this example.

Drawing a diagram allows you express the content of a problem in a non-verbal way. Often, you are able to simplify complex problems into basic components. As you draw your diagram, highlight the important parts of the question. The visual representation of the problem may contain clues to its solution that the verbal representation obscures.

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ☒ Visualization

6. Answer questions 49, 52.a., 52.b., 52.c., and 54 of "Applications and Problem Solving" on pages 200 and 201 of the textbook.

7. Turn to "TECHNOLOGY" on page 203 of *MATHPOWER™ 12* and answer the following:

- **a.** question 1 of Investigation 1, "The Nature of a Sine Curve"
- **b.** question 1 of Investigation 2, "The Nature of a Cosine Curve"
- **c.** question 1 of Investigation 3, "Comparing Sine and Cosine Curves"

8. Turn to page 200 of *MATHPOWER™ 12* and answer questions 35 to 42 of "Practice."

Turn to page 117 to compare your responses with those in the Appendix.

Now Try This

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ☒ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

In Investigation 3, "Comparing Sine and Cosine Curves" on page 203 of the textbook, you examined the similarity between sine and cosine and their repetitive patterns. Not all patterns are as simple to see as these.

9. The following sequence of numbers forms a pattern that is not simple to see, but is quite simple to explain when you have found the pattern generator. Determine the next number in the sequence, and explain how the sequence is derived.

11, 12, 13, 14, 15, 16, 17, 18, 21, 23, 25, 32, 101, …

Turn to page 122 to compare your response with the one in the Appendix.

If you have an integer sequence that you want to know more about, you might find it at the following website:

http://www.research.att.com/~njas/sequences

Looking Back

Mathematical Process

- ☒ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

In this activity, you defined the trigonometric ratios for arbitrary angles in standard position. You worked with these definitions to find exact and approximate values for the sine, cosine, tangent, secant, cosecant, and cotangent ratios.

Before you move to the next activity, explain how someone can tell if a trigonometric ratio is positive or negative in your journal. Include a list of real-world objects that have a repeating pattern in them that can be explained by trigonometry. Share your explanation with another student taking Pure Mathematics 30.

Activity 3: Graphing Sine and Cosine Functions

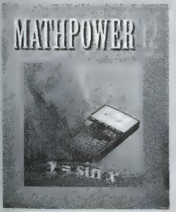
PHILIP JAMES CORWIN/CORBIS

Mathematical Process

- ■ Communication
- ☒ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

The big bass drums of a marching band provide the repetitive beat that helps the band to keep in time. Marching bands also rely on their drummer to give them cues when performing the intricate manoeuvres you see at football half-time shows. You can also hear repeating drum patterns in popular music, providing the driving beat for many top-ten hits. Do you think a song would be popular if there were no repeating patterns of rhythm or melody?

Repetitive patterns also occur in the graphs of sine and cosine functions. In this activity, you will graph sine and cosine functions and apply transformations to these graphs. In addition, you will analyse graphs of functions to determine their equations.



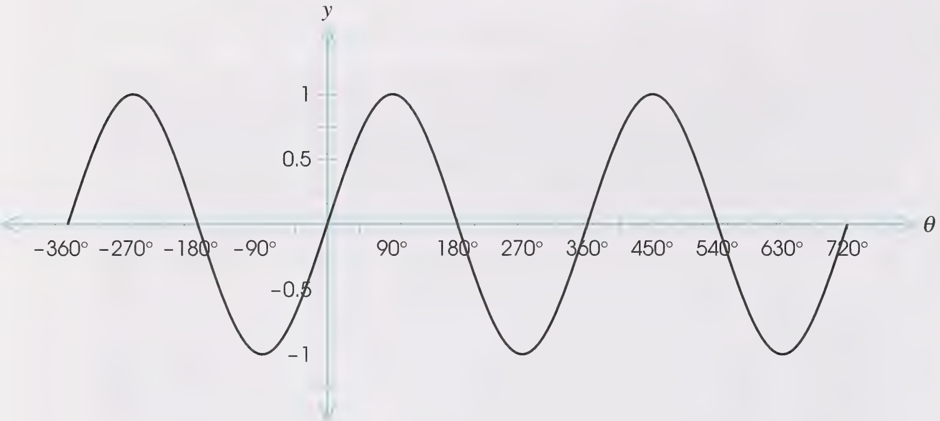
Turn to page 204 of *MATHPOWER™ 12* and read "Graphing Sine and Cosine Functions."

1. Answer the following questions on page 204 of the textbook:

  a. questions a. and b. of "Explore: Draw a Graph"
  b. questions 1 to 3 of "Inquire"

Turn to page 123 to compare your responses with those in the Appendix.

The graph you just drew of the sine function can be extended to the left for negative angles and to the right for angles greater than 360°.



The function $y = \sin\theta$ is an example of a periodic function, a function that repeats its values over a particular interval of its domain. Notice that the same values recur every 360°. This interval is called the period of this function.

Notice, too, that the graph extends one unit above and below the θ-axis as it repeats itself every 360°. This distance is called the amplitude of the graph.

Turn to page 205 in your textbook and read from the top of the page to Example 1.

To review these ideas, open the Videos/Animations folder on the Pure Mathematics 30 Companion CD and view the video titled *Periodic Events.*

a function that repeats its values over a particular interval of its domain

the smallest interval of the domain over which a periodic function repeats its values

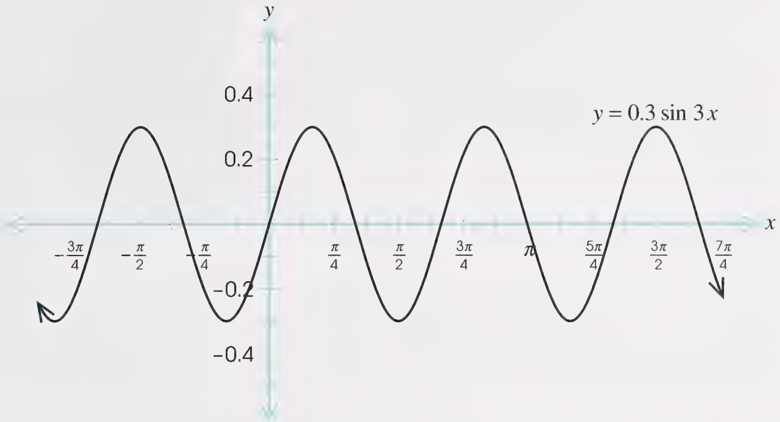
half the difference between the maximum and minimum values of a periodic function

Mathematical Process

■ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
☒ Visualization

Example

Find the amplitude and period of the following graph.



Solution

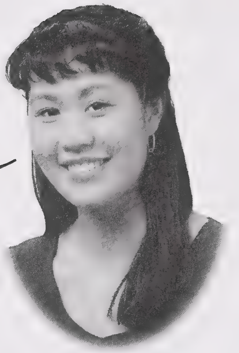
$$\begin{aligned}\text{amplitude} &= \frac{|\max - \min|}{2}\\ &= \frac{|0.3-(-0.3)|}{2}\\ &= \frac{0.6}{2}\\ &= 0.3\end{aligned}$$

Note: Amplitude is always positive. To ensure a positive value, absolute value is used in the formula. However, you can write the formula simply as amplitude $= \frac{\max - \min}{2}$.

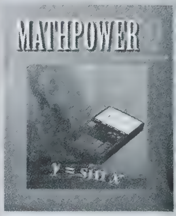
The period cannot be read directly from the graph, but notice that $1\frac{1}{2}$ periods of the graph fit into a space π wide. This allows you calculate the period.

$$\begin{aligned}\therefore\ \frac{3}{2}\text{period} &= \pi\\ \text{period} &= \frac{2}{3}\pi\end{aligned}$$

Now, it's your turn to find the period and amplitude of some periodic functions.



Turn to “INVESTIGATING MATH” on page 202 of *MATHPOWER™ 12* and read “Period and Amplitude.”



2. Answer question 1 of “Period and Amplitude.”

Turn to page 124 to compare your response with the one in the Appendix.

Turn to pages 205 and 206 of *MATHPOWER™ 12* and work through Examples 1 and 2. Notice the notation the textbook uses to describe the window settings for your calculator in Example 1.

You have seen examples of how the value of a in $y = a\sin x$ affects the graph of the function. Now, try a few questions yourself.

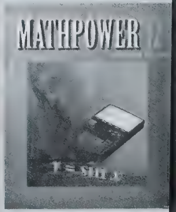
3. Answer questions 1, 4, 5, 9, and 10 of “Practice” on page 209 of the textbook.

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization

Turn to page 124 to compare your responses with those in the Appendix.

Turn to page 207 of *MATHPOWER™ 12* and read from the top of the page to the red line near the top of page 208, working through Examples 3 and 4.

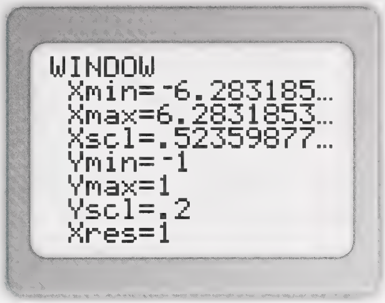


Example

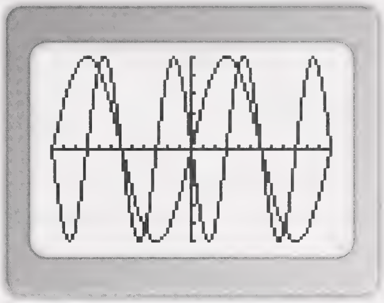
Use your graphing calculator to find the period (in radians) of the functions $y = 3\sin(-2\theta)$ and $y = -2.5\cos\frac{\theta}{2}$.

Solution

The functions $y = 3\sin(-2\theta)$ and $y = \sin(-2\theta)$ will have the same period. They will only differ in amplitude. Compare the graphs of $y = \sin\theta$ and $y = \sin(-2\theta)$ using the following window settings.

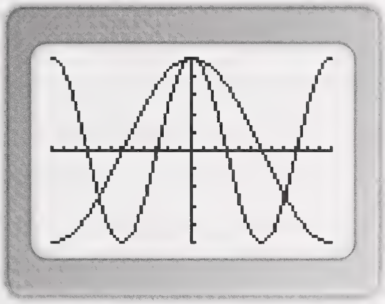


```
WINDOW
 Xmin=-6.283185…
 Xmax=6.2831853…
 Xscl=.52359877…
 Ymin=-1
 Ymax=1
 Yscl=.2
 Xres=1
```



From the graphs, the period of $y = \sin(-2\theta)$ is half the period of $y = \sin\theta$. Since the period of $y = \sin\theta$ is 2π, the period of $y = \sin(-2\theta)$, and thus of $y = 3\sin(-2\theta)$, is π (or $2\pi \div |-2|$).

The functions $y = \cos\frac{\theta}{2}$ and $y = -2.5\cos\frac{\theta}{2}$ will have the same period. Graph $y = \cos\theta$ and $y = \cos\frac{\theta}{2}$ using the preceding window settings.



From their graphs, the period of $y = \cos\frac{\theta}{2}$ is twice the period of $y = \cos\theta$. Since the period of $y = \cos\theta$ is 2π, the period of $y = \cos\frac{\theta}{2}$, and thus of $y = -2.5\cos\frac{\theta}{2}$, is 4π (or $2\pi \div |0.5|$).

4. Answer questions 12, 13, 14, 17, and 20 of "Practice" on page 209 of the textbook.

Turn to page 125 to compare your responses with those in the Appendix.

From the Pure Mathematics 30 Companion CD, open *The Trigonometric Functions Explorer* from the Explorers folder. Use this explorer to answer question 5.

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization

5. Use the graph of $y = \sin x$ to answer the following:

   a. Change the value of A from 1.0 to 1.5, then to 2.0, then to 0.6, then to 0.3, and then back to 1.0. How does changing A affect the graph?

   b. Change the value of B from 1.0 to 1.5, then to 2.0, then to 3.0, then to 0.5, and then back to 1.0. How does changing B affect the graph?

   c. Change the value of C from 0.0 to 0.5, then to 1.0, then to 1.5, then to 2.0, then to -0.5, then to -1.0, and then back to 0.0. How does changing C affect the graph?

   d. Change the value of D from 0.0 to 0.5, then to 1.0, then to -0.5, then to -1.0, and then back to 0.0. How does changing D affect the graph?

   e. Change the value of A from 1.0 to -1.0 and then from 1.5 to -1.5. How was the graph affected?

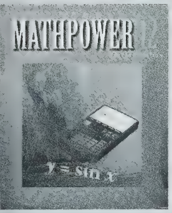
Turn to page 125 to compare your responses with those in the Appendix.

Turn to page 208 of *MATHPOWER™ 12* and work through Example 5. Pay close attention to the small note following the example. It summarizes the effects of coefficients a and b in the equation $y = a\sin(bx)$ or $y = a\cos(bx)$.

Remember: The amplitude is $|a|$, but the period is $\frac{2\pi}{|b|}$ or $\frac{360°}{|b|}$.

6. Answer questions 22, 23, 25, 28, and 29 of "Practice" on page 209 of the textbook.

Turn to page 130 to compare your responses with those in the Appendix.



Turn to pages 208 and 209 of *MATHPOWER™ 12* and work through Examples 6 and 7.

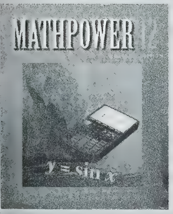
Now that you have discovered that coefficients *a* and *b* in equations of the form $y = a\sin bx$ affect the amplitude and period of their graphs, you will use this knowledge to solve problems that involve amplitude and period.

7. Answer the following questions on pages 209 to 211 of the textbook:

- **a.** questions 31, 35, 36, 39, 41, 43, 45, 48, and 50 of "Practice"
- **b.** questions 54, 55, 57, 58, 61, and 62 of "Applications and Problem Solving"

Turn to page 131 to compare your responses with those in the Appendix.

In Module 1, you studied transformations of graphs and functions. Now, you will apply your knowledge of transformations to the sine and cosine functions as you complete this activity.



Turn to page 212 of *MATHPOWER™ 12* and read "Further Transformations of Sine and Cosine Functions."

8. Answer the following questions on page 212 of the textbook:

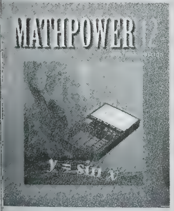
- **a.** questions a. to e. of "Explore: Analyze a Graph"
- **b.** questions 1 to 6 of "Inquire"

Turn to page 139 to compare your responses with those in the Appendix.

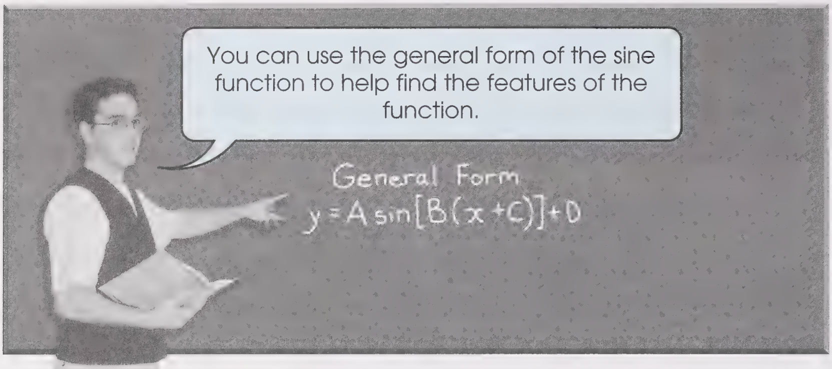
[illegible] a horizontal translation of a periodic function

In Module 1, you explored transformations of familiar functions. These transformations included vertical and horizontal sketches and vertical and horizontal translations. These same transformations can be applied to trigonometric functions. For trigonometric functions, a horizontal translation is often referred to as a [illegible].

You will now explore transformations of trigonometric functions in detail.

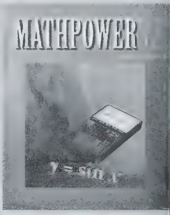


Turn to page 213 of *MATHPOWER™ 12* and read from the top of the page to Example 2 on page 214, working through Example 1.


You can use the general form of the sine function to help find the features of the function.
General Form
y = A sin[B(x + C)] + D

9. Answer questions 1, 3, 7, 8, 10, 13, and 15 of "Practice" on page 218 of the textbook.

Turn to page 141 to compare your responses with those in the Appendix.



Turn to page 214 of *MATHPOWER*™ *12* and work through Example 2. This example applies the principles involved in analysing complex sine functions. Notice the use of the calculator to find the *x*-intercepts.

You can also analyse a complex trigonometric function by using *The Trigonometric Functions Explorer* from the Explorers folder on the Pure Mathematics 30 Companion CD.

Mathematical Process
- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization

Example

For $y = 2\cos\left[\frac{3}{2}\left(x - \frac{\pi}{2}\right)\right] - 1.5$, use *The Trigonometric Functions Explorer* to determine the following:

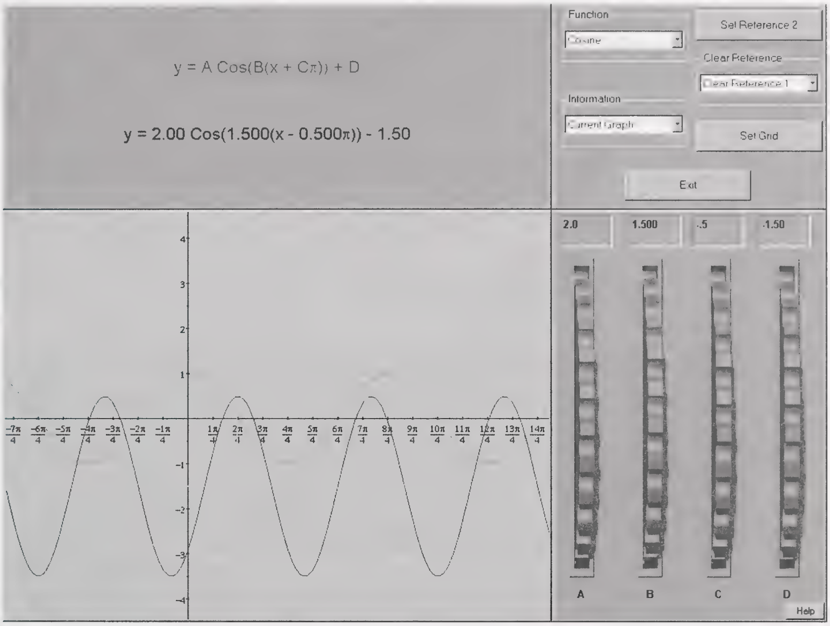
- **a.** domain and range
- **b.** period
- **c.** amplitude
- **d.** phase shift and vertical displacement
- **e.** y-intercept and x-intercepts, where $0 \le x \le 2\pi$

Solution

Step 1: Click on the "Set Grid" button, and set the domain to 2.0 and the range to 3.1.

Step 2: Choose the cosine function from the Function window, and set the reference function to $y = \cos x$ by clicking the "Set Reference 1" button.

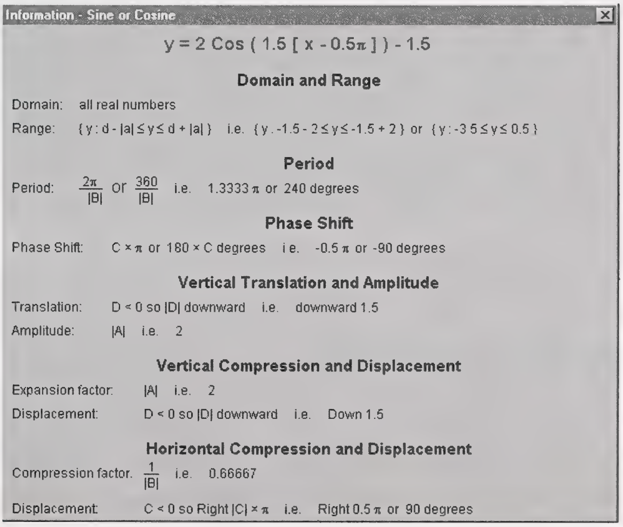
Step 3: Click on the "Set Function" button and set the parameters to $A = 2$, $B = 1.5$, $C = -0.50$, and $D = -1.5$.



y = A Cos(B(x + Cπ)) + D

y = 2.00 Cos(1.500(x - 0.500π)) - 1.50

Step 4: Choose "Current Graph" from the Information window. The following window appears.



Information - Sine or Cosine

y = 2 Cos (1.5 [x - 0.5π]) - 1.5

Domain and Range

Domain: all real numbers

Range: { y : d - |a| ≤ y ≤ d + |a| } i.e. { y : -1.5 - 2 ≤ y ≤ -1.5 + 2 } or { y : -3.5 ≤ y ≤ 0.5 }

Period

Period: $\frac{2\pi}{|B|}$ or $\frac{360}{|B|}$ i.e. 1.3333π or 240 degrees

Phase Shift

Phase Shift: C × π or 180 × C degrees i.e. -0.5π or -90 degrees

Vertical Translation and Amplitude

Translation: D < 0 so |D| downward i.e. downward 1.5

Amplitude: |A| i.e. 2

Vertical Compression and Displacement

Expansion factor: |A| i.e. 2

Displacement: D < 0 so |D| downward i.e. Down 1.5

Horizontal Compression and Displacement

Compression factor: $\frac{1}{|B|}$ i.e. 0.66667

Displacement: C < 0 so Right |C| × π i.e. Right 0.5π or 90 degrees

Use the Information window to answer questions a. to d.

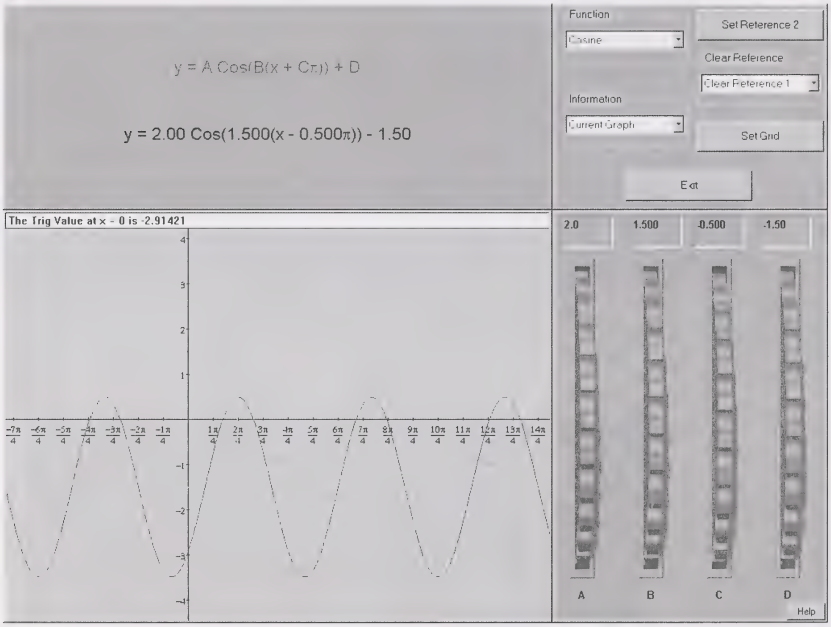
a. The domain is the set of real numbers, and the range is $-3.5 \leq y \leq 0.5$.

b. The period is $\frac{4}{3}\pi$.

c. The amplitude is 2.

d. The phase shift is 0.5π to the right, and the vertical displacement is 1.5 units down.

e. To approximate the x- and y-intercepts, move the cursor to one of the intercepts and click while pressing the shift key.



y = A Cos(B(x + Cπ)) + D

y = 2.00 Cos(1.500(x - 0.500π)) - 1.50

The Trig Value at x - 0 is -2.91421

Function: Cosine

Information: Current Graph

Clear Reference: Clear Reference 1

A: 2.0

B: 1.500

C: -0.500

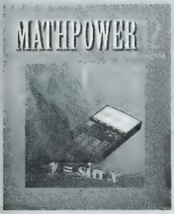
D: -1.50

The y-intercept is approximately -2.91. The x-intercepts within $0 \leq x \leq 2\pi$ are approximately 1.09, 2.05, 5.27, and 6.23.

From the Pure Mathematics 30 Companion CD, open the Computer Assisted Instruction folder and work through the segments in *Graphing Basic Trigonometric Functions*. For now, only work through the segments involving $y = \sin \theta$, $y = \cos \theta$, and $y = \tan \theta$.

Graphs that result from the transformation of $y = \sin x$ are called sinusoidal graphs. The graphs of $y = \cos x$ and its transformations are also considered sinusoidal. To discover why, read on.

shaped like a sine curve



Turn to pages 215 to 218 of *MATHPOWER*™ *12* and work through Examples 3 to 5.

10. Answer the following questions on pages 218 and 219 of the textbook:

a. questions 17, 19, 23, 24, 26, 27, 29, 31, and 32 of "Practice"
b. questions 34, 35, 37, and 38 of "Applications and Problem Solving"

Turn to page 142 to compare your responses with those in the Appendix.

Now Try This

Mathematical Process
☐ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☒ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

You may find it easy to do calculations involving trigonometric functions because you have a calculator with values available to you at the press of a few buttons. This was not possible 300 or 400 years ago. The development of special ways to find values to use in computations was a major part of the work done by mathematicians of that time.

Here's a little problem that shows the kind of work mathematicians 300 or 400 years ago had to work on.

11. An acute angle, A, has a tangent of $\frac{1}{2}$. A second acute angle, B, has a tangent of $\frac{1}{3}$. A third acute angle, C, has a tangent of 1. Draw a diagram to show that the sum of the measures of angles A, B, and C is $\frac{\pi}{2}$.

Turn to page 153 to compare your response with the one in the Appendix.

Looking Back

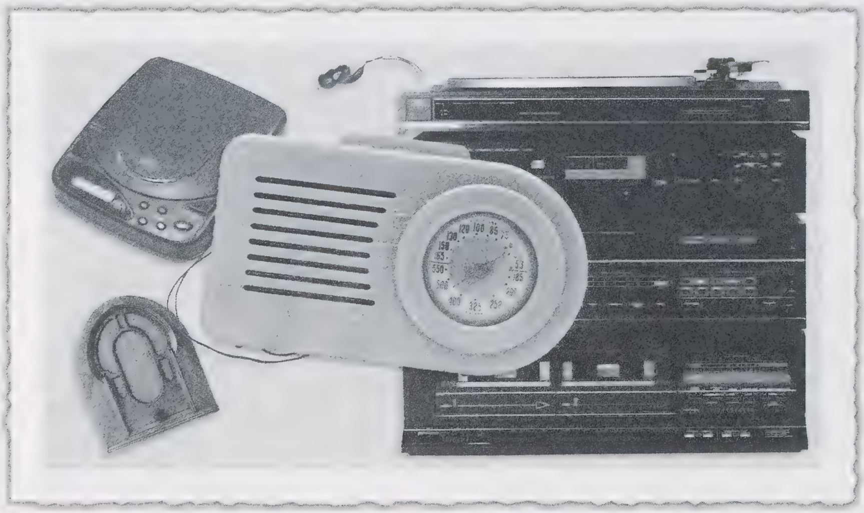
In this activity, you studied the definitions of the trigonometric ratios and their transformations. You examined how the coefficients—*A*, *B*, *C*, and *D*—of the general sine and cosine functions, $y = A \sin B(x + C) + D$ or $y = A \cos B(x + C) + D$, affect their graphs. You also looked at the reverse relationship—how to build the equation from a graph by determining the values for *A*, *B*, *C*, and *D*. You have seen that the trigonometric functions allow you to solve problems involving circles, angles, and even events in the environment.

In your journal, write a concise summary of the things you have learned about graphing trigonometric functions. Pay particular attention to the effect of the $B(x + C)$ part of the general sine and cosine functions.

Mathematical Process

- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Activity 4: Applications of Sinusoidal Functions

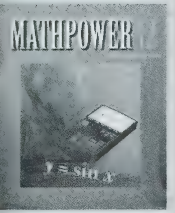


When you sit back and enjoy your favourite music, it probably never occurs to you that sine waves are working very hard to enhance the music. Each of the devices in the preceding picture uses carefully controlled periodic waves to reproduce the music. The radio has to have an extremely accurate frequency or you will not be able to tune in the station. The CD player has to output exactly 44.1 kHz or the music will not be what the artists intended. Even the cassette player has a high-frequency bias wave (about 100 kHz) to help make the music match the artists' intentions.

For more information about how electronic devices (like cassette players, CD players, and radios) work, visit the following website:

http://www.howstuffworks.com/category-electronics.htm

In this activity, you will be using sine and cosine to analyse real-world situations and solve problems.

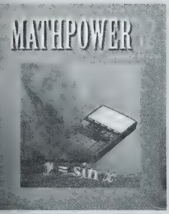


Turn to page 222 of *MATHPOWER™ 12* and read “Applications of Sinusoidal Functions.”

1. Answer the following questions on page 222 of the textbook:

   a. “Explore: Draw a Graph”
   b. questions 1 to 6 of “Inquire”

Turn to page 154 to compare your responses with those in the Appendix.



Turn to page 222 of *MATHPOWER™ 12* and read from the red line to the red line on page 225, working through Examples 1 to 3. In these examples the parameters A and B are assumed to be positive to obtain simpler forms of the function.

2. Answer the following questions on pages 225 to 227 of the textbook:

   a. questions 1 to 3 and 6 to 15 of “Practice”
   b. questions 17, 19, 21, and 24.a. of “Applications and Problem Solving”

Turn to page 157 to compare your responses with those in the Appendix.

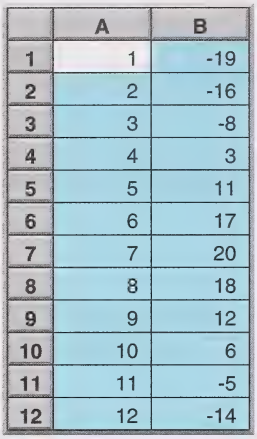
You can also use
The Trigonometric Functions Explorer
to map a graph onto a set of data points.
This explorer allows you to plot data points
from a spreadsheet or from a text editor,
such as *Notepad* or *SimpleText*,
onto the graph window.

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☒ Technology
- ☐ Visualization

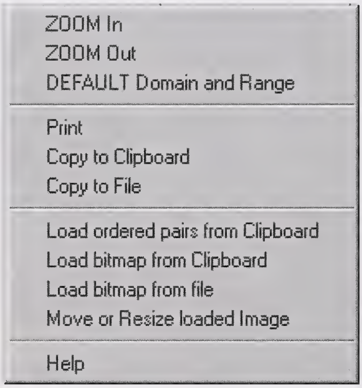
The following steps show how to use *The Trigonometric Functions Explorer* to map a graph to the data points given in Example 1 on page 223 of the textbook.

Step 1: Enter the data into a spreadsheet. Then select the data points you want to place into the explorer, and copy them.

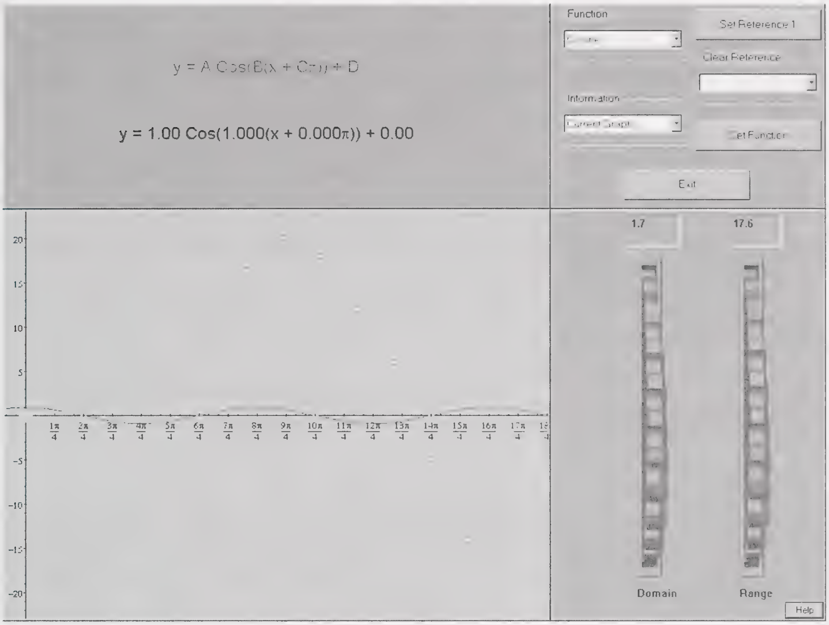


	A	B
1	1	-19
2	2	-16
3	3	-8
4	4	3
5	5	11
6	6	17
7	7	20
8	8	18
9	9	12
10	10	6
11	11	-5
12	12	-14

Step 2: Open *The Trigonometric Functions Explorer*, and right-click the graph window. The following menu will appear.



Step 3: Choose “Load ordered pairs from Clipboard.” The data points will be placed in the graph window. **Note:** You may have to adjust the domain and range in order to see the data points. You may also wish to drag the coordinate axes to the left. The display should be similar to the following.



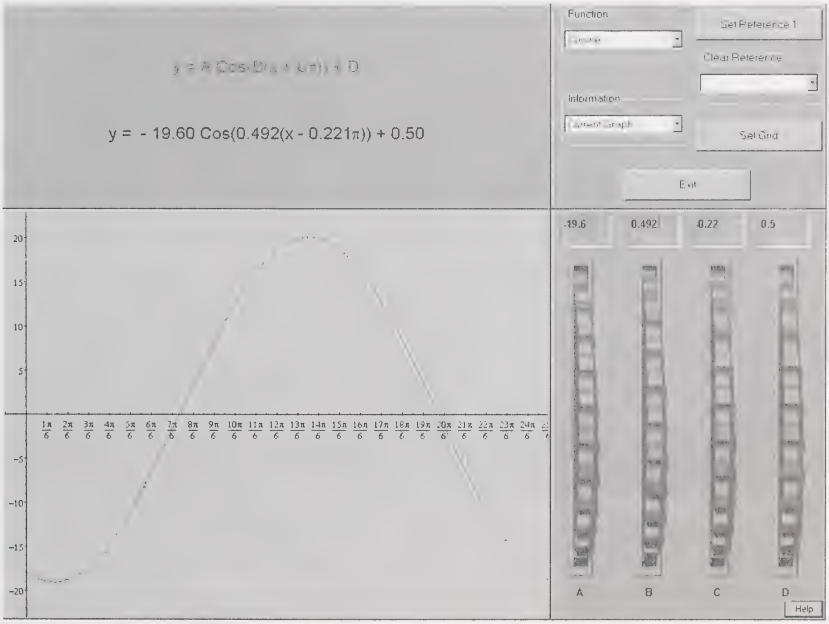
y = A Cos(B(x + Cπ)) + D

y = 1.00 Cos(1.000(x + 0.000π)) + 0.00

Domain: 1.7

Range: 17.6

Step 4: Using the coefficients of the equation, adjust the graph until it best fits the data points. The display should be similar to the following.



y = A Cos(B(x + Cπ)) + D

y = - 19.60 Cos(0.492(x - 0.221π)) + 0.50

A: -19.6

B: 0.492

C: -0.22

D: 0.5

Function
Set Reference 1
Clear Reference
Information
Current Graph
Set Grid
Exit
20
15
10
5
-5
-10
-15
-20
1π/6 2π/6 3π/6 4π/6 5π/6 6π/6 7π/6 8π/6 9π/6 10π/6 11π/6 12π/6 13π/6 14π/6 15π/6 16π/6 17π/6 18π/6 19π/6 20π/6 21π/6 22π/6 23π/6 24π/6
Help

The equation is approximately $y = -19.6 \cos 0.492\left(x - 0.221\pi\right) + 0.50$.

3. **a.** Using *The Trigonometric Functions Explorer* from the Pure Mathematics 30 Companion CD, load the data from the file *WinnipegTemp.txt* into the graph window and fit a sine curve to the data. This file is found in the Module_Work folder.

   b. Test your model equation by using it to predict temperatures for March, May, and October (months 3, 5, and 10). Is your model for the temperature accurate?

Turn to page 164 to compare your responses with those in the Appendix.

The Trigonometric Functions Explorer also allows you to map sinusoidal curves to objects in a picture. If you have access to a graphic or photo that can be mapped by a sinusoidal curve, experiment with this feature on the explorer.

Note: To move and resize the graphic, right-click the graph window and choose "Move or Resize Loaded Image." When you are finished resizing the graphic and wish to manipulate the curve, simply double-click the graph window.

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☒ Reasoning
- ☐ Technology
- ☐ Visualization

Now Try This

In this activity, you have seen several examples of functions that have identical graphs but are represented by different equations. Functions like $y = \sin x$ and $y = \cos\left(x - \frac{\pi}{2}\right)$ is one example. Their graphs match exactly, but their equations appear different. You've seen alternate names for the same object many of times before. For example you can name the number 3 in many of ways, such as three, $1 + 2$, III, and $6 - 3$.

4. How many ways can you name 100 using the following restrictions?

- You must use 1, 2, 3, 4, 5, 6, 7, 8, 9 each exactly once.
- The digits must be in the order 1, 2, 3, 4, 5, 6, 7, 8, 9.
- You can use the operations + or – between digits as often as you wish.

Turn to page 166 to compare your response with the one in the Appendix.

You can obtain information regarding question 4 and many others like it at the following website:

http://www.cut-the-knot.com/do_you_know/digits.html

Mathematical Process

- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

Looking Back

In this activity, you applied sinusoidal functions to practical problems. You saw that many phenomena in nature roughly follow a sinusoidal pattern. However, finding the equation of the sine or cosine function that best fits the data can be difficult without the aid of technology.

In your journal, explain how to find the amplitude, period, phase shift, and vertical displacement for a given sinusoidal graph. Include in your explanation how your graphing calculator can be used to help find some of these values when a practical problem, such as Example 2 on page 224 of *MATHPOWER™ 12*, needs to be solved.

Activity 5: Other Trigonometric Functions

When you study subjects in school, it can be easy to forget that they often have many practical applications. Trigonometry is such a subject. Its applications are found in some surprising places. In your school, perhaps hidden just a ceiling tile or two away, are electrical cables enclosed in metal conduits. If you follow a run of conduit, you will see it curve to pass over obstacles or branch to meet junction boxes. The photo displays a variety of conduits leading from one part of the building to another. These conduits contain bends and angles that can be calculated using trigonometric functions, such as the secant function.

For more information about offset bends and how trigonometry is used in figuring them out, visit the following website:

http://www.porcupinepress.com/_bending/offsets.htm

Turn to page 228 of *MATHPOWER*™ *12* and read “Other Trigonometric Functions.”

1. Answer the following questions on page 228 of the textbook:

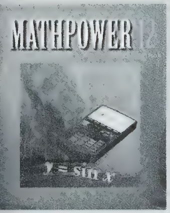
a. “Explore: Use Skills With Functions”

b. questions 1 to 8 of “Inquire”

Turn to page 166 to compare your responses with those in the Appendix.

Mathematical Process

■ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
☒ Technology
■ Visualization



When you were using your graphing calculator to verify predictions in questions 5 to 7 of "Inquire," you may have noticed a number of vertical lines appearing along with the graphs of tangent functions. These vertical lines are **asymptotes** of the tangent functions and appear at $\frac{\pi}{2}+k\pi$, where k is any integer. The domain of the tangent function excludes these values. These values, where tangent is not defined, are the same values where cosine is 0.

Turn to page 229 of *MATHPOWER*™ *12* and read from the top of the page to the bottom of page 231, working through Examples 1 to 3. In Example 1, notice how transformations affect the domain, range, and period. The changes in the domain are changes you did not have to worry about when you worked with the sine and cosine functions.

2. Answer questions 1, 2, 4, 7, 9, and 10 of "Practice" on page 232 of the textbook.

3. Use *The Trigonometric Functions Explorer* from the Pure Mathematics 30 Companion CD to answer the following:

  a. Graph $y=\tan x$. Change the value of A from 1.0 to 2.0, then to 3.0, then to 0.5, and then to 0.3. How does changing the value of A affect the graph?

  b. Graph $y=\cot x$. Change the value of B from 1.0 to 2.0, then to 3.0, then to 0.5, and then to 0.1. How does changing the value of B affect the graph?

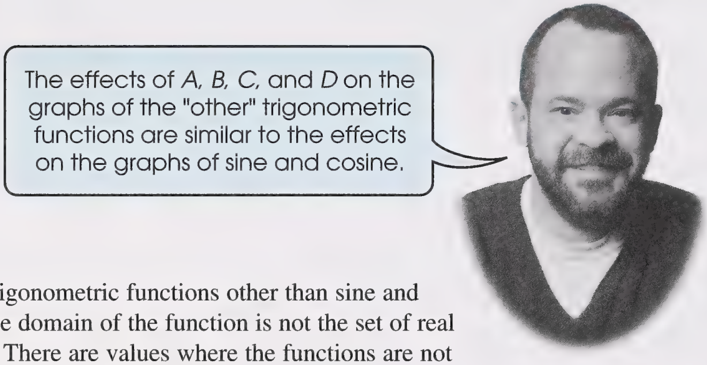
  c. Graph $y=\sec x$. Change the value of C from 0.0 to 0.5, then to 1.0, then to 1.5, then to -0.5, and then to -1.0. How does changing the value of C affect the graph?

  d. Graph $y=\csc x$. Change the value of D from 0.0 to 0.5, then to 1.0, then to -0.5, and then to -1.0. How does changing the value of D affect the graph?

Turn to page 169 to compare your responses with those in the Appendix.

You can generalize from your experimental results. For a function defined as $y = A \text{ trig } B(x + C) + D$, where trig is either tangent, secant, cosecant, or cotangent, the following statements are true:

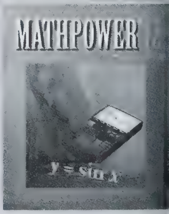
- *If* $|A|$ *increases, the graph stretches vertically.*
- *If* $|B|$ *increases, the period of the graph decreases, thus making the excluded values in the domain move closer together. If* $|B|$ *decreases, the period of the graph increases, thus making the excluded values in the domain move further apart.*
- *If C increases, the graph moves to the left. If C decreases, the graph moves to the right. This change is the phase shift or horizontal translation.*
- *If D increases, the graph moves upward. If D decreases, the graph moves downward. This change is the vertical translation.*


The effects of *A*, *B*, *C*, and *D* on the graphs of the "other" trigonometric functions are similar to the effects on the graphs of sine and cosine.

For the trigonometric functions other than sine and cosine, the domain of the function is not the set of real numbers. There are values where the functions are not defined. You will notice that your graphing calculator will draw vertical lines at the values that are excluded from the domains of these functions.

4. Turn to page 232 of *MATHPOWER*™ *12* and answer questions 16, 17, 19, 21, and 25 of "Applications and Problem Solving."

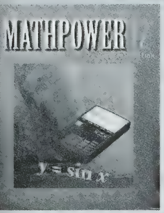
5. Turn to "INVESTIGATING MATH" on page 233 of *MATHPOWER*™ *12* and answer questions 4 and 6 of "Absolute Value and Trigonometric Functions."



Turn to page 173 to compare your responses with those in the Appendix.

Mathematical Process

■ Communication
■ Connection
■ Estimation
■ Mental Math
☒ Problem Solving
■ Reasoning
■ Technology
■ Visualization



Now Try This

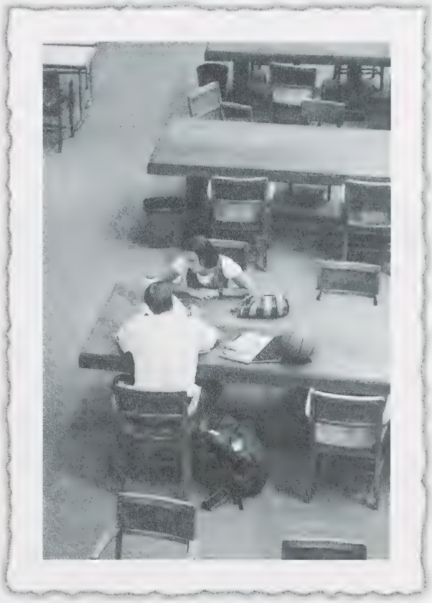
Some mathematicians describe mathematics as a study of patterns. These mathematicians are always looking for the way objects relate to one another. Often, finding these relationships requires considerable creativity. In the last activity, you saw that $\frac{\pi}{2}$ was the same as the sum of the angles with tangents of $\frac{1}{2}$, $\frac{1}{3}$, and 1. The following problem, regarding relationships, also requires a bit of creativity and logical thinking to solve.

6. Turn to page 201 of *MATHPOWER™ 12* and answer "LOGIC POWER."

Turn to page 179 to compare your response with the one in the Appendix.

Looking Back

In this activity, you worked with graphs of the "other" trigonometric functions. You saw that their graphs have "holes" in them that correspond to values that are not in the domains of the functions. You then discovered how to graph the secant, cosecant, tangent, and cotangent functions when various transformations have been applied; and you solved problems involving these trigonometric functions.



To review the graphs of tangent, cotangent, secant, and cosecant, open the Computer Assisted Instruction folder from the Pure Mathematics 30 Companion CD and work through the segments in *Graphing Basic Trigonometric Functions*.

Mathematical Process

☒ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
☒ Technology
■ Visualization

In your journal, write an explanation of how to find the period, phase shift, and vertical displacement for a given tangent, secant, cosecant, or cotangent graph. Also, explain how the domains of these functions change as the values of coefficients *B* and *C* change. Compare your explanations with those of another student taking Pure Mathematics 30.

Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

The intent of the activities in this section is to give you a firm understanding of trigonometric functions and how changing the values of the coefficients in their equations affects the graphs. One of the more obvious changes in the graph of a periodic function is the change in period. You have seen that for a function of the form $y = A \sin B(x + C) + D$, the **period** is related to the coefficient ***B***.

Example

Determine the period of the following functions.

a. $y = -17 \cos 3(x + \pi) - 199$

b. $y = 13 \tan\left[-6(x + 3)\right] - 12$

c. $y = 5 \sec 0.5(x - 1.08) + 2.065$

Solution

a. Compare $y = -17 \cos 3(x + \pi) - 199$ with $y = A \cos B(x + C) + D$.
Therefore, $B = 3$.

$$\begin{aligned} \therefore \text{ Period} &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{|3|} \\ &= \frac{2\pi}{3} \end{aligned}$$

b. Compare $y = 13\tan\left[-6(x+3)\right] - 12$ with $y = A\tan B(x+C) + D$.

Therefore, $B = -6$.

$$\begin{aligned} \therefore \text{ Period} &= \frac{\pi}{|B|} \\ &= \frac{\pi}{|-6|} \\ &= \frac{\pi}{6} \end{aligned}$$

c. Compare $y = 5\sec 0.5(x - 1.08) + 2.065$ with $y = A\sec B(x+C) + D$.

Therefore, $B = 0.5$.

$$\begin{aligned} \therefore \text{ Period} &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{|0.5|} \\ &= 4\pi \end{aligned}$$

From the preceding examples, notice how the coefficient B relates to the period. Also, notice that the period of a tangent function (and cotangent function) uses a different formula than the other four trigonometric functions.

Period Formula for Tangent and Cotangent Functions

$$\text{Period} = \frac{\pi}{|B|}$$

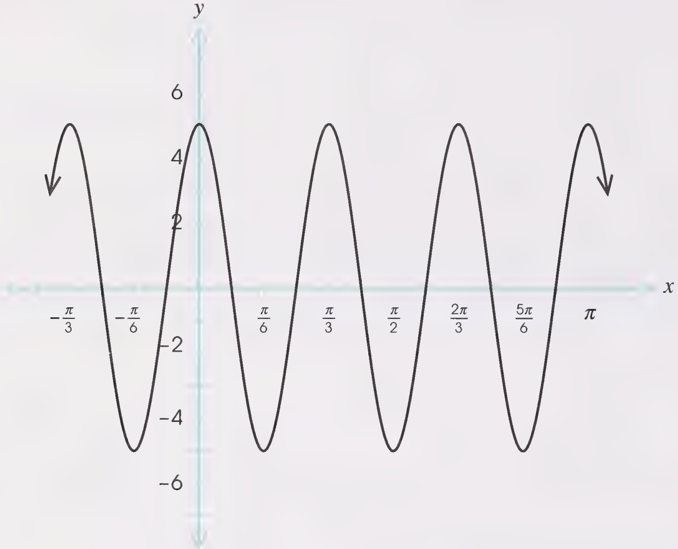
Period Formula for the Other Trigonometric Functions

$$P = \frac{2\pi}{|B|}$$

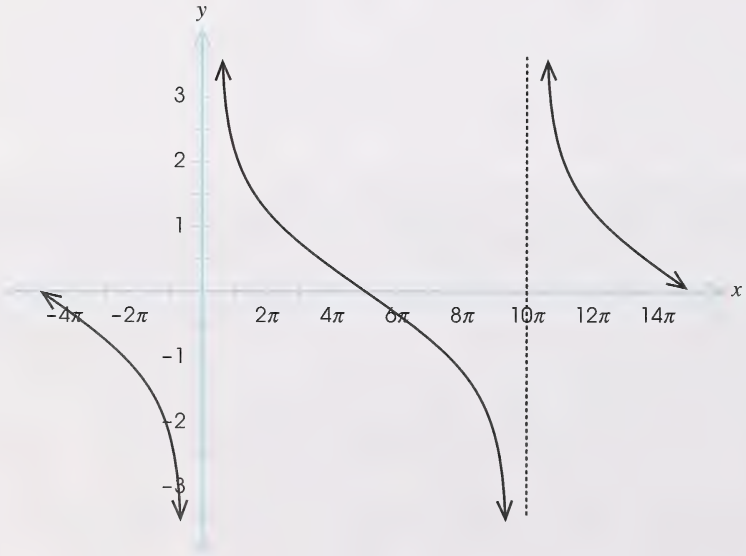
Example

Determine the period of the following graphs.

a.



b.



Solution

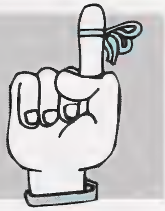
a. The graph has one maximum at $x = 0$. The next maximum occurs at $x = \frac{\pi}{3}$.

$$\begin{aligned}\therefore \text{ Period} &= \frac{\pi}{3} - 0 \\ &= \frac{\pi}{3}\end{aligned}$$

b. One vertical asymptote occurs at $x = 0$. The next vertical asymptote occurs at $x = 10\pi$.

$$\begin{aligned}\therefore \text{ Period} &= 10\pi - 0 \\ &= 10\pi\end{aligned}$$

To find the period from a graph, you need to find the difference between x-values at the start and at the end of one period. It may be easier to use maximum values, minimum values, or zero values of the function in locating starting and ending points of a period.



Another obvious change in a graph is a phase shift (or horizontal translation). Unless the phase shift is very close to the period of the graph, you can generally see that the graph has moved horizontally. The **phase shift** is related to the coefficient C in the general form for trigonometric functions, $y = A \text{ trig } B(x + C) + D$, where trig is one of sine, cosine, tangent, cotangent, secant, or cosecant. If $C > 0$ the graph shifts to the left. If $C < 0$ the graph shifts to the right.

Example

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization

Determine the phase shift for the following functions. Check your answers using your graphing calculator.

a. $y = \sec\left(x + \frac{\pi}{2}\right)$

b. $y = 3\sec\left[-4\left(x - \frac{2\pi}{3}\right)\right]$

c. $y = -5\sec 1.7(x+3) - 4$

Solution

a. Compare $y = \sec\left(x + \frac{\pi}{2}\right)$ with $y = A\sec B(x+C) + D$.

Because $C = \frac{\pi}{2}$, the phase shift is $\frac{\pi}{2}$ to the left.

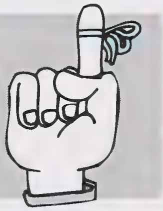
b. Compare $y = 3\sec\left[-4\left(x - \frac{2\pi}{3}\right)\right]$ with $y = A\sec B(x+C) + D$.

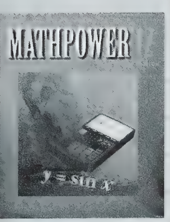
Because $C = -\frac{2\pi}{3}$, the phase shift is $\frac{2\pi}{3}$ to the right.

c. Compare $y = -5\sec 1.7(x+3) - 4$ with $y = A\sec B(x+C) + D$.

Because $C = 3$, the phase shift is 3 units to the left.

To find the phase shift of a graph, determine the *x*-value that starts one period of the translated graph and compare it to the same starting point on the standard graph. You may find it easier to use maximum values, minimum values, or zero values.





1. Turn to pages 236 and 237 of *MATHPOWER™ 12* and answer questions 32 to 43 of "Review."

Turn to page 182 to compare your responses with those in the Appendix.

For the non-sinusoidal trigonometric functions—tangent, cotangent, secant, and cosecant—the domain is not the entire set of real numbers. There are excluded values, or "holes," in the domains of these functions.

Example

Determine the domain of the following functions.

- **a.** $y = \csc x$
- **b.** $y = \cot x + 4$
- **c.** $y = \sec 3x$
- **d.** $y = \tan\left(2x + \frac{\pi}{2}\right) + 4$

Solution

a. Because $y = \csc x = \frac{1}{\sin x}$, the function is undefined when $\sin x = 0$. (Division by zero is undefined.) It is these values that are excluded from the domain of the cosecant function. Therefore, the domain of $y = \csc x$ is the set of real numbers except for $k\pi$, where k is any integer.

b. Because $y = \cot x = \frac{\cos x}{\sin x}$, the function is undefined when $\sin x = 0$. (Division by zero is undefined.) It is these values that are excluded from the domain of the cotangent function. Therefore, the domain of $y = \cot x + 4$ is the set of real numbers except for $k\pi$, where k is any integer. **Note:** The vertical translation of 4 units up does not affect the domain; it affects the range.

c. Because $y = \sec 3x = \frac{1}{\cos 3x}$, the function is undefined when $\cos 3x = 0$. It is these values that are excluded from the domain of $\sec 3x$.

Because $\cos x = 0$ when $x = \frac{\pi}{2} + k\pi$, where k is any integer, $\cos 3x = 0$ when

$$\begin{aligned} 3x &= \frac{\pi}{2} + k\pi \\ x &= \frac{\pi}{6} + \frac{k\pi}{3} \end{aligned}$$

Therefore, the domain of $y = \sec 3x$ is the set of real numbers except for $\frac{\pi}{6} + \frac{k\pi}{3}$, where k is any integer.

d. $$\begin{aligned} y &= \tan\left(2x + \frac{\pi}{2}\right) + 4 \\ &= \tan 2\left(x + \frac{\pi}{4}\right) + 4 \end{aligned}$$

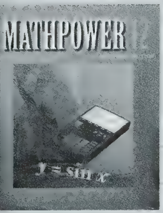
Because $\tan 2x = \frac{\sin 2x}{\cos 2x}$, the function is undefined when $\cos 2x = 0$. It is these values that are excluded from the domain of $\tan 2x$.

Because $\cos x = 0$ when $x = \frac{\pi}{2} + k\pi$, where k is any integer, $\cos 2x = 0$ when

$$\begin{aligned} 2x &= \frac{\pi}{2} + k\pi \\ x &= \frac{\pi}{4} + \frac{k\pi}{2} \end{aligned}$$

The graph of $y = \tan 2\left(x + \frac{\pi}{4}\right) + 4$ is the graph of $y = \tan 2x$ translated $\frac{\pi}{4}$ to the left and 4 units up. Because the phase shift affects the "holes," the domain of $y = \tan\left(2x + \frac{\pi}{2}\right) + 4$ is the set of real numbers except for

$$\begin{aligned} \frac{\pi}{4} + \frac{k\pi}{2} - \frac{\pi}{4} &= 0 + \frac{k\pi}{2} \\ &= \frac{k\pi}{2}, \text{ where } k \text{ is any integer} \end{aligned}$$



2. Turn to page 237 of *MATHPOWER™ 12* and answer questions 50 and 51 of "Review."

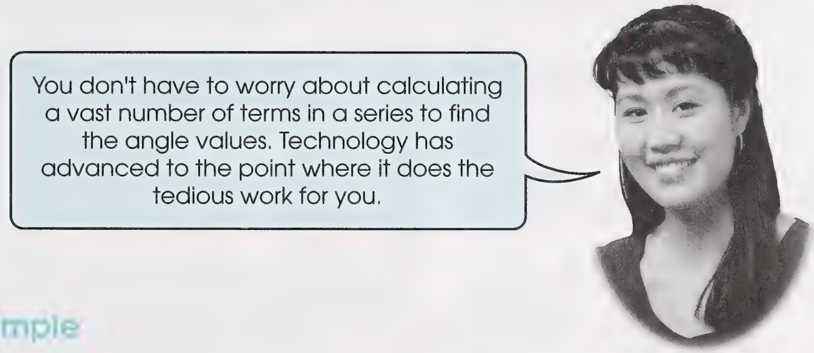
Turn to page 187 to compare your responses with those in the Appendix.

Enrichment

In Activity 4, you were asked to solve a problem about a paddle steamer. As part of your solution, you had to find the angle that made a sinusoidal function zero. This aspect of the problem had occupied mathematicians for several hundred years. It was a question that involved the inverse functions. These functions are often called arcsine, arccosine, and arctangent. On your graphing calculator, they appear as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$. This notation does not imply taking a negative exponent, but rather represents the inverse function.

There is an interesting discussion about finding angles from values of the tangent function at the following website:

http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibpi.html#greg


You don't have to worry about calculating a vast number of terms in a series to find the angle values. Technology has advanced to the point where it does the tedious work for you.

Example

Determine the angles that have the following tangents.

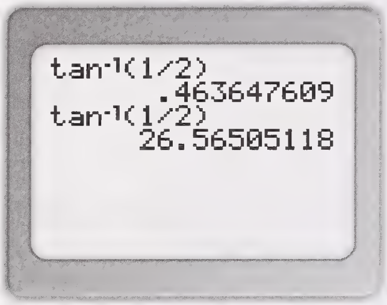
a. $\frac{1}{2}$ **b.** $\frac{1}{3}$ **c.** $\frac{2}{3}$

d. 2 **e.** -1.5 **f.** 1

Solution

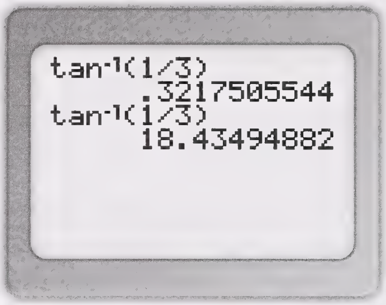
The following displays gives the answer in radians first and degrees second.

a.



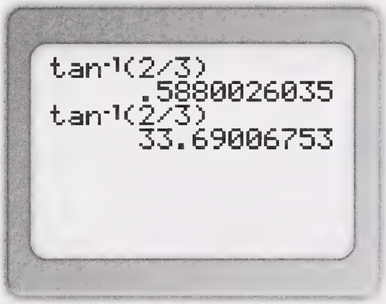
```
tan⁻¹(1/2)
          .463647609
tan⁻¹(1/2)
          26.56505118
```

b.



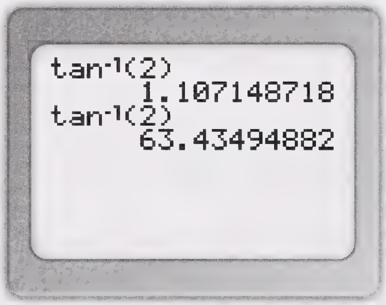
```
tan⁻¹(1/3)
          .3217505544
tan⁻¹(1/3)
          18.43494882
```

c.



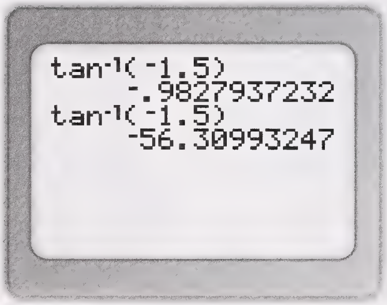
```
tan⁻¹(2/3)
          .5880026035
tan⁻¹(2/3)
          33.69006753
```

d.



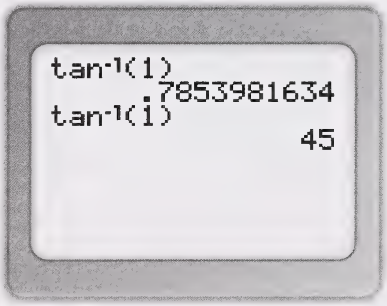
```
tan⁻¹(2)
          1.107148718
tan⁻¹(2)
          63.43494882
```

e.



```
tan⁻¹(-1.5)
         -.9827937232
tan⁻¹(-1.5)
         -56.30993247
```

f.



```
tan⁻¹(1)
          .7853981634
tan⁻¹(1)
                   45
```

Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization

Example

Determine the angle, x, associated with the following trigonometric values.

a. $\sin x = \frac{1}{3}$ **b.** $\cos x = \frac{2}{3}$ **c.** $\sin x = \frac{2}{3}$

d. $\cos x = 0.123\ 25$ **e.** $\tan x = 0.011\ 24$ **f.** $\sin x = 0.011\ 24$

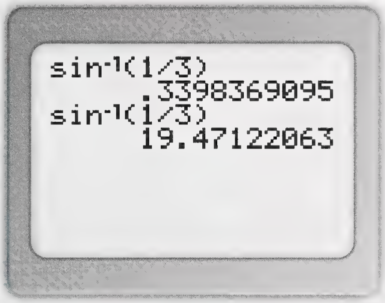
Mathematical Process

- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ■ Visualization

Solution

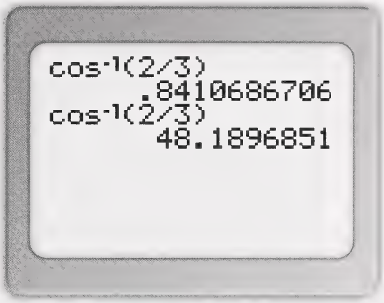
The following displays give the answer in radians first and degrees second.

a.



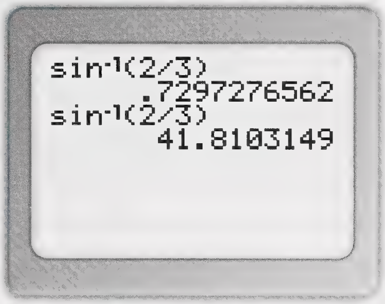
```
sin⁻¹(1/3)
    .3398369095
sin⁻¹(1/3)
    19.47122063
```

b.



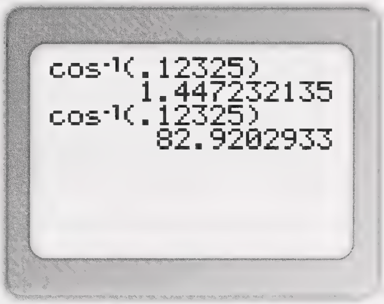
```
cos⁻¹(2/3)
    .8410686706
cos⁻¹(2/3)
      48.1896851
```

c.



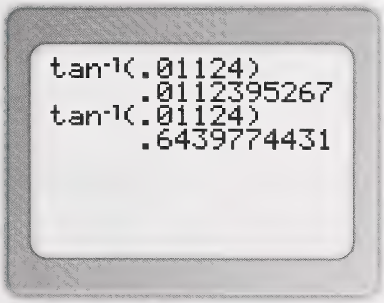
```
sin⁻¹(2/3)
    .7297276562
sin⁻¹(2/3)
      41.8103149
```

d.



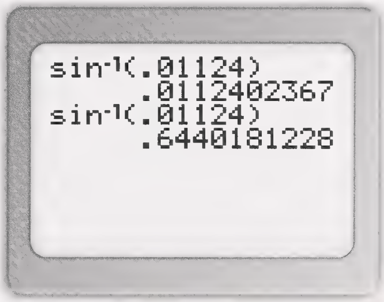
```
cos⁻¹(.12325)
    1.447232135
cos⁻¹(.12325)
      82.9202933
```

e.



```
tan⁻¹(.01124)
    .0112395267
tan⁻¹(.01124)
    .6439774431
```

f.



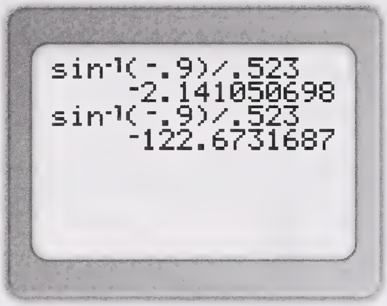
```
sin⁻¹(.01124)
    .0112402367
sin⁻¹(.01124)
    .6440181228
```

Example

For the function $y = 2\sin 0.523x + 1.8$, find one x-intercept.

Solution

$$\begin{aligned}
y &= 2\sin 0.523x + 1.8 \\
0 &= 2\sin 0.523x + 1.8 \\
2\sin 0.523x &= -1.8 \\
\sin 0.523x &= \frac{-1.8}{2} \\
\sin 0.523x &= -0.9 \\
\sin^{-1}(\sin 0.523x) &= \sin^{-1}(-0.9) \\
0.523x &= \sin^{-1}(-0.9) \\
x &= \frac{\sin^{-1}(-0.9)}{0.523} \\
&\doteq -2.14 \text{ rad or } -122.7^\circ
\end{aligned}$$



```
sin⁻¹(-.9)/.523
         -2.141050698
sin⁻¹(-.9)/.523
         -122.6731687
```

1. Calculate an approximate zero of the function $y = 6\sin 0.5x - 1.8$.
2. Calculate an approximate zero of the function $y = -2\cos\frac{\pi}{6}x + \frac{9}{5}$.

Turn to page 188 to compare your responses with those in the Appendix.

Conclusion

In this section, you explored the relationship between radian measure and degree measure. You then used radians and degrees, within the context of angles in standard position, to explore the primary trigonometric functions—sine, cosine, and tangent—and the reciprocal trigonometric functions—secant, cosecant, and cotangent. You examined the graphs of these functions and analysed the relationships among their graphs and equations. An integral part of your exploration was applying trigonometric functions to model periodic events.

You discovered that periodic events (such as the tides, the phases of the moon, wave motion, a swinging pendulum, and the position of riders on a Ferris wheel) can be modelled using the various forms of the sine and cosine functions. You can use your models to analyse these events, to discover new relationships, and to make predictions. Creating and using models is an important aspect of the work "applied mathematicians" (such as physicists, economists, and engineers) perform.

Assignment

Turn to Assignment Booklet 5A and complete the assignment for Section 1. Remember to submit your completed Assignment Booklet for assessment before starting Section 2.

Section 2

THE PURCELL TEAM/CORBIS

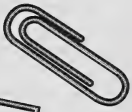
Trigonometric Equations

The Bluenose has been a part of Canadian folklore since the 1920s. This schooner was designed to carry cargo and to race. Did you know that it won every race it entered? You can find an image of the Bluenose on the "tails" side of a Canadian dime.

Merchant sailing vessels have carried goods from port to port for thousands of years. With modern navigational aids, like GPS (global positioning system), knowing where you are and what direction to sail is fairly easy. Sailors at the turn of the first millennium didn't have such technological wonders. They had to use more mundane methods to find their way, like sailing along the shore and watching for landmarks. More adventurous sailors used the positions of the stars along with some trigonometry to determine their location. Charts and maps were developed to help sailors find their way.

What happened if a storm blew you off course? How might the captain of a vessel in the first millennium determine the ship's location and determine the direction to shore? Problems in navigation were overcome with the development of more sophisticated instruments and mathematical tools. Trigonometry became an integral part of this process.

In this section, you will solve equations involving trigonometric functions. You will use your graphing calculator and your knowledge of the basic trigonometric functions to work your way through the activities of this section. You will then discover some very interesting facts about equivalencies among trigonometric expressions, and you will use them to solve complex problems.



Activity 1: Relating Graphs and Solutions

A picture can get your mind working in ways that surprise everyone. A solution to the problem on this student's mind is found in the following recipe.

Chocolate Cake

- 1 egg
- 1 cup sugar
- $\frac{1}{2}$ cup shortening
- $\frac{1}{2}$ cup milk
- $\frac{1}{2}$ teaspoon vanilla
- 1 teaspoon salt
- $1\frac{1}{2}$ cups flour
- $\frac{1}{2}$ cup cocoa
- 1 teaspoon baking soda
- $\frac{3}{4}$ cup boiling water

Mix the ingredients together, and pour the mixture into a round pan. Bake for 30 min at 375°F.

The picture and the solution are not related directly to trigonometry, but the result can be almost as rewarding. You will use pictures to solve trigonometric problems. In this activity, you will use graphs of trigonometric functions to find solutions to trigonometric equations.

Turn to page 244 of *MATHPOWER™ 12* and read “Relating Graphs and Solutions.”

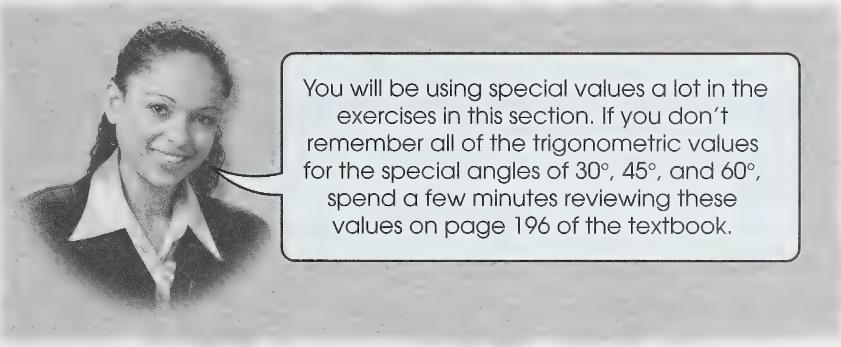
1. Answer the following questions on page 244 of the textbook:

- **a.** questions a. and b. of “Explore: Use a Graph”
- **b.** questions 1 to 6 of “Inquire”

Turn to page 189 to compare your responses with those in the Appendix.

From the Pure Mathematics 30 Companion CD, open the Computer Assisted Instruction folder and choose *Graphing Basic Trigonometric Functions*. Work through the segments you feel you need to review. A good review of sketching trigonometric functions will assist you in solving problems involving trigonometric equations.

Turn to page 245 of *MATHPOWER™ 12* and work through Example 1. Notice that the values of $\sin x$ for the special angles 0 and π are used in the solution to the example.



You will be using special values a lot in the exercises in this section. If you don’t remember all of the trigonometric values for the special angles of 30°, 45°, and 60°, spend a few minutes reviewing these values on page 196 of the textbook.

2. Answer questions 2, 5, and 10 of “Practice” on pages 247 and 248 of the textbook.

Turn to page 189 to compare your responses with those in the Appendix.

Next, you will explore a trigonometric equation involving a reciprocal trigonometric function.

Example

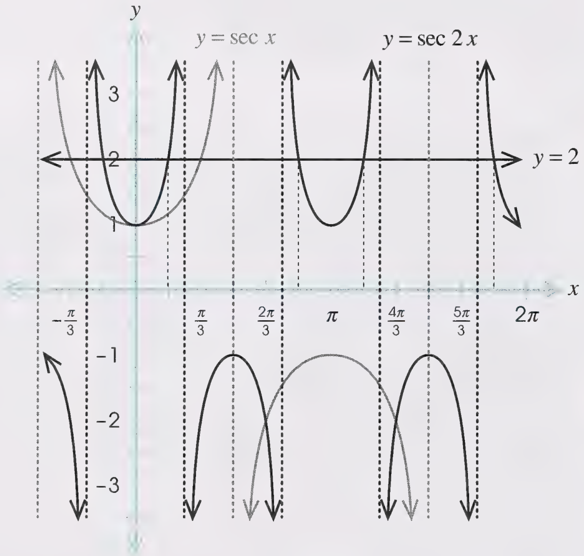
Sketch the graph of $y = \sec 2x$. Use the graph to find the solutions of $\sec 2x = 2$ in the interval $0 < x < 2\pi$.

Solution

The graph of $y = \sec 2x$ is related to the graph of $y = \sec x$ by a horizontal compression factor of 2. In the interval $0 < x < 2\pi$, $y = \sec x$ has one period; therefore, $y = \sec 2x$ has two periods.

The period of $y = \sec 2x$ is $\frac{2\pi}{2}$ or π.

The solutions to the equation $\sec 2x = 2$ are equivalent to finding the x-values on the graph that have a corresponding y-value of 2 in the specified interval. From the graph, these x-values are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$.



Therefore, the solutions to $\sec 2x = 2$ in the interval $0 < x < 2\pi$ are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$.

Mathematical Process

■ Communication
☒ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
☒ Visualization

In the preceding examples, you explored how to find the solutions to a trigonometric equation for a given interval on the x-axis, such as $0 \le x \le 360°$. If no restriction is given, how would you determine all solutions?

Example

Solve $\sin x = 1$ for the following intervals.

a. $0 \le x < 2\pi$

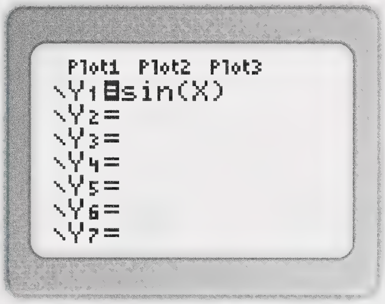
b. $x \in R$

Mathematical Process

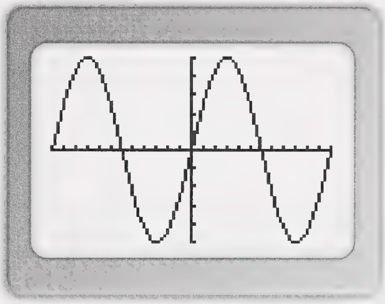
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☑ Technology
- ☐ Visualization

Solution

Graph $y = \sin x$ on your graphing calculator.

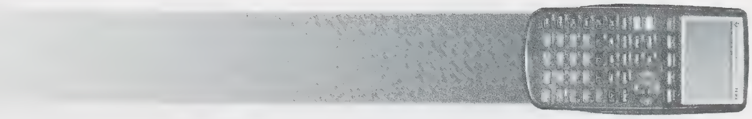


```
Plot1 Plot2 Plot3
\Y1=sin(X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



a. In the interval $0 \le x < 2\pi$, $\sin x = 1$ when $x = \frac{\pi}{2}$. Therefore, the solution to $\sin x = 1$ in the interval $0 \le x < 2\pi$ is $\frac{\pi}{2}$.

b. The interval $0 \le x < 2\pi$ is just one period of the function $y = \sin x$. Therefore, if x is not restricted to $0 \le x < 2\pi$, the complete set of solutions is $x = \frac{\pi}{2} + 2n\pi$, where n is any integer.



The solution to question b. in the preceding example is often referred to as the general solution, the solution when no restriction on the domain of the trigonometric function is given.

the set of all solutions in the domain of the trigonometric functio appearing in the equation

To find the general solution of a trigonometric equation, first find all solutions to the trigonometric equation for one period of the function. Then extend the solutions to the rest of the domain by adding or subtracting multiples of the period from each of the solutions obtained in the first step.

Turn to pages 245 and 246 of *MATHPOWER™ 12* and work through Example 2.

3. Answer questions 3, 9, 12, and 15 of "Practice" on pages 247 and 248 of the textbook.

4. Find all the solutions to the equation $\csc 3x = -\frac{2\sqrt{3}}{3}$ in the interval $0 < x < 2\pi$.

Turn to page 190 to compare your responses with those in the Appendix.

You need to combine your knowledge of special angles and trigonometric function values with the power of a graphing calculator to solve some problems.

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☒ Technology
- ☐ Visualization

Turn to pages 246 and 247 of *MATHPOWER™ 12* and work through Examples 3 and 4.

5. Answer the following questions on page 248 of the textbook:

   a. questions 16, 18, and 20 of "Practice"
   b. questions 21, 24, 26, 27, and 29 of "Applications and Problem Solving"

Turn to page 193 to compare your responses with those in the Appendix.

Now Try This

Mathematical Process
☐ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☒ Problem Solving
☒ Reasoning
☒ Technology
☐ Visualization

Throughout this module, technology (such as the graphing calculator) has helped you solve problems. These problems could have been solved without these aids most of the time, but it would take a great deal more time.

6. Here is a little addition problem for you to solve. Try it without using your calculator.

Each letter in the following addition problem represents a unique digit (0, 1, 2, 3, 4, 5, 6, 7, 8, or 9), where the left-most letter of each word is not 0. What is the value of each letter?

$$\begin{array}{r} \text{FIVE} \\ \text{FIVE} \\ \text{FIVE} \\ +\ \text{ELEVEN} \\ \hline \text{THIRTY} \end{array}$$

Turn to page 197 to compare your response with the one in the Appendix.

Looking Back

Mathematical Process
☒ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

In this activity, you worked with first-degree trigonometric equations. You discovered how graphing or sketching the equation's parts can assist in finding solutions. You also studied the special values of the trigonometric functions. Finally, you applied your knowledge of the period of trigonometric functions to find the general solutions to trigonometric equations.

In your journal, explain how the period of a trigonometric function is used to find the general solutions to a first-degree trigonometric function. Compare your explanation with that of another student taking Pure Mathematics 30.

Activity 2: Solving Trigonometric Equations

When possible, to save fuel pilots fly along the shortest flight paths to their destinations. These flight paths are curved. Because Earth is spherical, planes fly along arcs or "great circles" above the surface. A straight line may be the shortest distance between two points; however, you cannot fly along this path unless you can fly through the Earth.

You can find the lengths of flight paths using a little trigonometry and solving trigonometric equations. In this activity, you will solve trigonometric equations to answer practical problems.

Mathematical Process

- ■ Communication
- ☒ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

Turn to page 249 of *MATHPOWER*™ *12* and read "Solving Trigonometric Equations."

1. Answer the following questions on page 249 of the textbook:

- **a.** "Explore: Use a Graph"
- **b.** questions 1 to 5 of "Inquire"

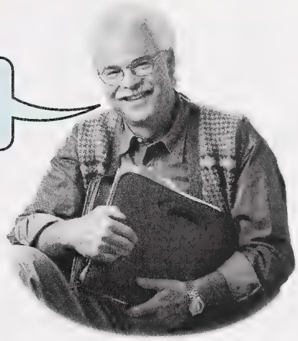
Turn to page 198 to compare your responses with those in the Appendix.

In the preceding questions, you used both a graphical and an algebraic approach to solve trigonometric equations. This activity will give you extended practice in melding algebra, graphical approaches, and technology to solve equations.

Turn to pages 249 and 250 of *MATHPOWER™ 12* and work through Example 1.

2. Answer questions 3, 4, 6, 10, 11, 17, 18, 19, and 20 of "Practice" on page 252 of the textbook.

Turn to page 199 to compare your responses with those in the Appendix.



A number of trigonometric equations are similar in form to quadratic equations.

Example

Convert each of the following quadratic functions into a product of linear terms.

a. $f(x)=x^2+x$ and $F(x)=\sin^2 x+\sin x$

b. $g(x)=x^2+2x+1$ and $G(x)=\tan^2 x+2\tan x+1$

Solution

a.
$$\begin{aligned} f(x)&=x^2+x \\ &=(x)(x+1) \end{aligned}$$

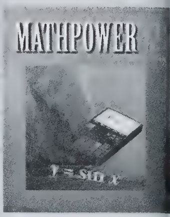
$$\begin{aligned} F(x)&=\sin^2 x+\sin x \\ &=(\sin x)(\sin x+1) \end{aligned}$$

b.
$$\begin{aligned} g(x)&=x^2+2x+1 \\ &=(x+1)(x+1) \end{aligned}$$

$$\begin{aligned} G(x)&=\tan^2 x+2\tan x+1 \\ &=(\tan x+1)(\tan x+1) \end{aligned}$$

The preceding example introduces you to the idea of factoring trigonometric equations. The next example shows how factoring will assist you in finding solutions to second-degree equations.

Turn to pages 250 and 251 of *MATHPOWER™ 12* and work through Example 2.



If the quadratic equation cannot be factored over the integers, you will have to use the quadratic formula.

Example

Rewrite each of the following quadratic functions as a product of linear factors.

a. $f(z)=z^2+3z+1$ and $F(x)=\sin^2 x+3\sin x+1$

b. $g(z)=z^2+2z-1$ and $G(x)=\tan^2 x+2\tan x-1$

Solution

a. The equation $f(z)=z^2+3z+1$ does not factor into linear factors containing integral coefficients only. The factors must be found by using the quadratic formula.

$$\begin{aligned}\therefore z&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\&=\frac{-3\pm\sqrt{3^2+4(1)(1)}}{2(1)}\\&=\frac{-3\pm\sqrt{9-4}}{2}\\&=\frac{-3\pm\sqrt{5}}{2}\end{aligned}$$

$$\therefore f(z)=\left(z-\frac{-3+\sqrt{5}}{2}\right)\left(z-\frac{-3-\sqrt{5}}{2}\right)$$

Notice that the two equations have the same form: one involves z and the other $\sin x$. This comparison yields the following result:

$$\begin{aligned}F(x)&=\sin^2 x+3\sin x+1\\&=\left(\sin x-\frac{-3+\sqrt{5}}{2}\right)\left(\sin x-\frac{-3-\sqrt{5}}{2}\right)\end{aligned}$$

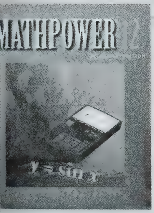
b. The equation $g(z) = z^2 + 2z - 1$ does not factor over the integers. The factors will have to be found using the quadratic formula.

$$\begin{aligned} \therefore\ z &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 \pm \sqrt{2^2 - (4)(1)(-1)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{4+4}}{2} \\ &= \frac{-2 \pm 2\sqrt{2}}{2} \\ &= -1 \pm \sqrt{2} \end{aligned}$$

$$\therefore\ g(z) = \left(z + 1 - \sqrt{2}\right)\left(z + 1 + \sqrt{2}\right)$$

Notice that the two equations have the same form: one involves z and the other $\tan x$. This comparison yields the following result:

$$\begin{aligned} G(x) &= \tan^2 x + 2 \tan x - 1 \\ &= \left(\tan x + 1 - \sqrt{2}\right)\left(\tan x + 1 + \sqrt{2}\right) \end{aligned}$$



Turn to page 251 of *MATHPOWER™ 12* and work through Example 3.

3. Answer questions 21 and 22 of “Practice,” on page 252 of the textbook.

Turn to page 206 to compare your responses with those in the Appendix.

Example

Solve $\cos^2 x - 0.25 = 0$ for *x*, where $0 \le x < 2\pi$; then give a general solution.

Solution

The equation can be solved by writing $\cos^2 x - 0.25 = 0$ as a product of linear factors.

$$\begin{aligned} \cos^2 x - 0.25 &= 0 \\ (\cos x + 0.5)(\cos x - 0.5) &= 0 \end{aligned}$$

$$\therefore \ \cos x + 0.5 = 0 \quad \text{or} \quad \cos x - 0.5 = 0$$

$$\cos x = -0.5 \qquad\qquad \cos x = 0.5$$

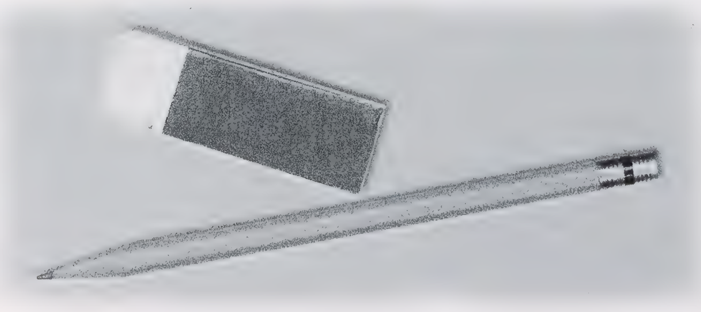
$$x = \frac{2\pi}{3} \text{ and } \frac{4\pi}{3} \qquad\qquad x = \frac{\pi}{3} \text{ and } \frac{5\pi}{3}$$

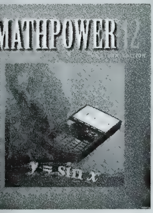
The solutions to $\cos^2 x - 0.25 = 0$ in the interval $0 \le x < 2\pi$ are $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, and $\frac{5\pi}{3}$.

The general solution is found by adding multiples of the period of the function, 2π, to the solution. Therefore, the general solutions for $\cos^2 x - 0.25 = 0$ are $x = \frac{\pi}{3} + 2k\pi$, $x = \frac{2\pi}{3} + 2k\pi$, $x = \frac{4\pi}{3} + 2k\pi$, and $x = \frac{5\pi}{3} + 2k\pi$, where *k* is any integer.

Notice that $x = \frac{\pi}{3} + 2k\pi$ and $\frac{4\pi}{3} + 2k\pi$ can be combined as $x = \frac{\pi}{3} + k\pi$, where *k* is any integer. Also, notice that $x = \frac{2\pi}{3} + 2k\pi$ and $x = \frac{5\pi}{3} + 2k\pi$ can be combined as $x = \frac{2\pi}{3} + k\pi$, where *k* is any integer.

Therefore, the four general forms can be combined as $\theta = \frac{\pi}{3} + k\pi$ and $\theta = \frac{2\pi}{3} + k\pi$, where *k* is any integer.





Turn to page 252 of *MATHPOWER™ 12* and work through Example 4.

4. Answer the following questions on pages 252 and 253 of the textbook:

- **a.** questions 25, 29, and 30 of "Practice"
- **b.** questions 33, 35, 36, and 40 of "Applications and Problem Solving"

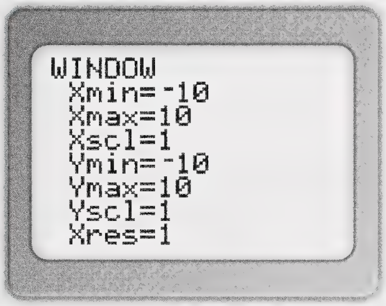
Turn to page 207 to compare your responses with those in the Appendix.

Now Try This

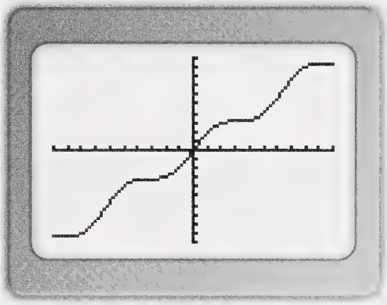
athematical
rocess
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

You have just spent some time using graphs to help solve problems. In mathematics, graphs are often used to make information simpler to see. In some cases, though, the information needed may be hidden within the complexities of the graph.

5. The following graph shows a trigonometric function. Use the graph to determine the equation of this function.



```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
```



Turn to page 216 to compare your response with the one in the Appendix.

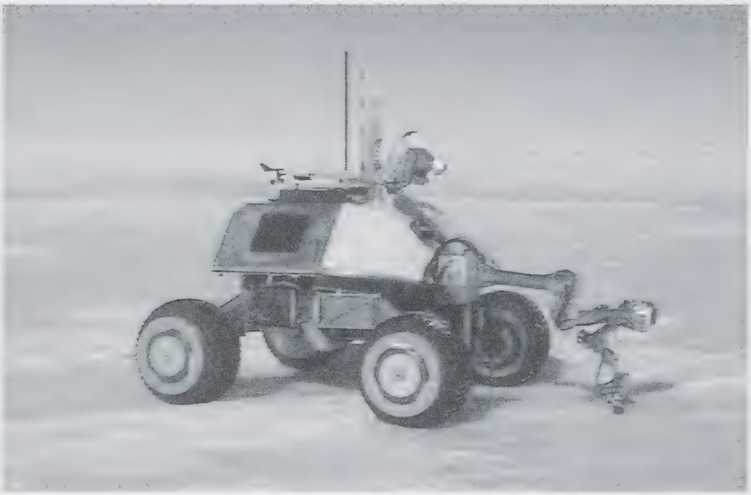
Looking Back

athematical
rocess
- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization

In this activity, you solved trigonometric equations and applied these techniques to solve practical problems. You combined graphical and algebraic methods to obtain specific and general solutions to both first- and second-degree trigonometric equations.

In your journal, explain how to solve second-degree trigonometric equations. Be sure to explain how a graphing calculator makes these solutions easier to find.

Activity 3: Using Technology


CARNEGIE MELLON UNIVERSITY AND NASA

Mathematical Process
■ Communication
☒ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

The Nomad, used in the Robotic Antarctic Meteorite Search, is conducting an automated search for meteorites in Antarctica. It has a pair of cameras to give it a three-dimensional view like the three-dimensional vision humans have. Differences between the images from the two cameras give the Nomad depth perception. If you know the distance between the cameras and the angles through which they are rotated, you can use trigonometry to calculate the distance an object is from the Nomad. This is just a small part of the mathematics involved in moving the arm in the front of the Nomad to examine a possible meteorite.

Technology allows you to perform tasks that would be difficult or otherwise impossible. In this activity, you will use technology to help you solve trigonometric equations.

Turn to page 254 of *MATHPOWER™ 12* and read "Using Technology to Solve More Complex Equations."

1. Answer the following questions on page 254 of the textbook:

   a. questions a. and b. of "Explore: Use a Graph"
   b. questions 1 to 4 of "Inquire"

Turn to page 217 to compare your responses with those in the Appendix.

Mathematical Process

- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☒ Problem Solving
- ☐ Reasoning
- ☒ Technology
- ☐ Visualization

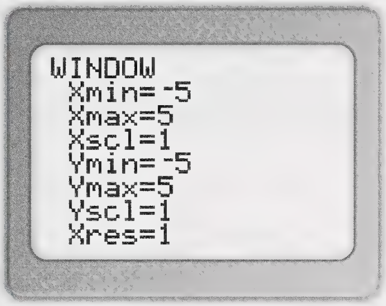
Example

Use your graphing calculator to solve the equation $2\cos x = x + 1$. Round your answers to 4 decimal places.

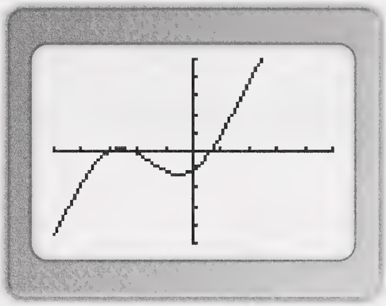
Solution

This equation has a graph that has been shifted 1 unit up, resulting in more solutions than the textbook example has. The solution to this question is the same as the solution to $x - 2\cos x + 1 = 0$.

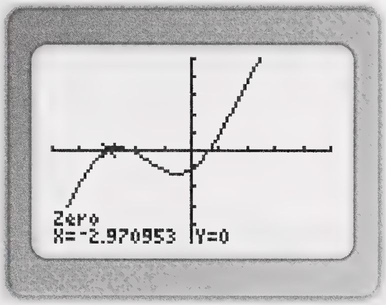
Graph $y = x - 2\cos x + 1$ on your graphing calculator.



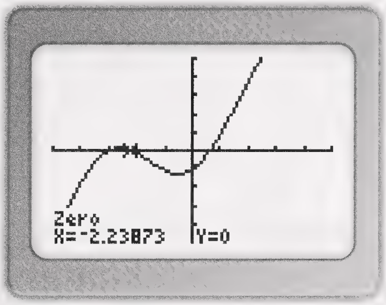
```
WINDOW
 Xmin=-5
 Xmax=5
 Xscl=1
 Ymin=-5
 Ymax=5
 Yscl=1
 Xres=1
```



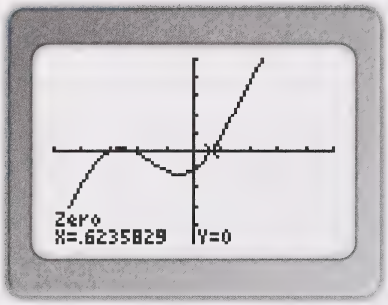
Use the Zero feature to find the *x*-intercepts.



```
Zero
X=-2.970953  Y=0
```



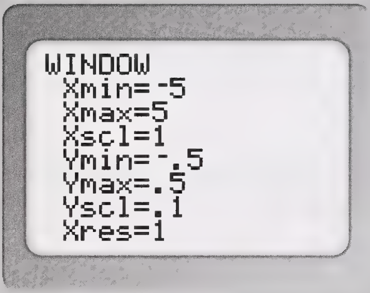
```
Zero
X=-2.23873   Y=0
```



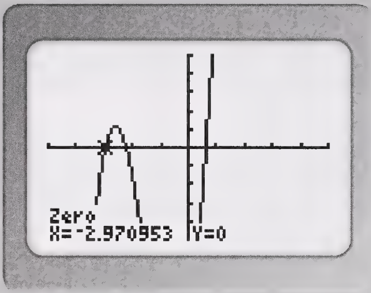
```
Zero
X=.6235829    Y=0
```

The x-intercepts are approximately -2.9710, -2.2387, and 0.6236.

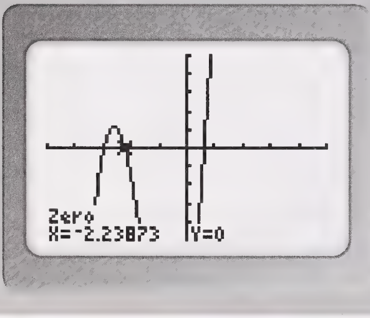
If you use a smaller range of y-values, you can more easily distinguish between the two zeros on the left.



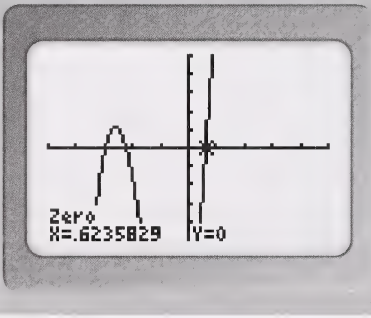
```
WINDOW
 Xmin=⁻5
 Xmax=5
 Xscl=1
 Ymin=⁻.5
 Ymax=.5
 Yscl=.1
 Xres=1
```



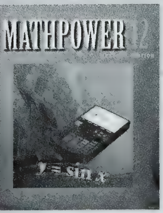
```
Zero
X=⁻2.970953   Y=0
```



```
Zero
X=⁻2.23873    Y=0
```



```
Zero
X=.6235829    Y=0
```



2. Turn to pages 256 and 257 of *MATHPOWER™ 12* and answer questions 1 to 7 of "Practice."

Turn to page 219 to compare your responses with those in the Appendix.

Next, you will apply your newly acquired skills to solve practical problems.

Turn to page 256 of *MATHPOWER™ 12* and work through Example 3.

Mathematical Process

■ Communication
■ Connection
■ Estimation
■ Mental Math
☒ Problem Solving
☒ Reasoning
☒ Technology
■ Visualization

Example

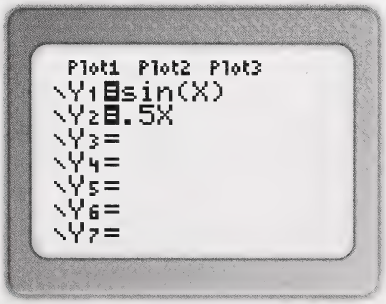
a. Graph $y = \sin x$ and $y = 0.5x$ in the interval $-\pi \le x \le \pi$.

b. Use the graph to solve the equation $\sin x = 0.5x$ to 3 decimal places. Did the graph provide all of the solutions to the equation? Explain.

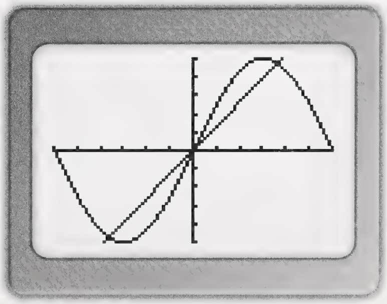
c. Would the equation $|\sin x| = 0.5x$ have the same roots as in question b.? Explain.

Solution

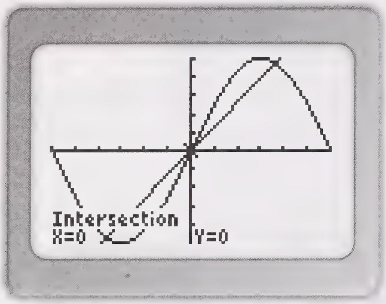
a.



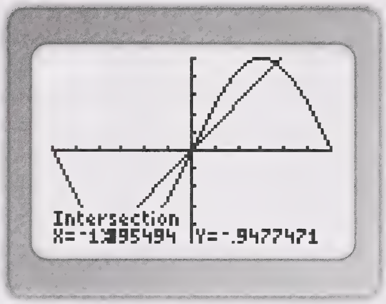
```
Plot1 Plot2 Plot3
\Y1=sin(X)
\Y2=.5X
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



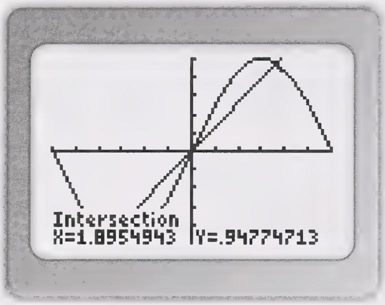
b.



Intersection
X=0 Y=0



Intersection
X=-1.895494 Y=-.9477471



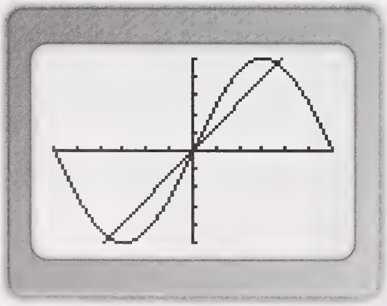
Intersection
X=1.8954943 Y=.94774713

The solutions to $\sin x = 0.5x$ are 0 and ± 1.895, correct to three decimal places.

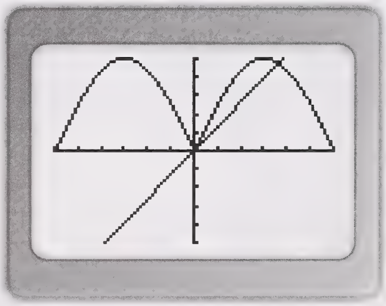
Because the graph of $y = 0.5x$ extends beyond the range of $\sin x$ outside the interval $-\pi \le x \le \pi$, all solutions to $\sin x = 0.5x$ are shown in the graph.

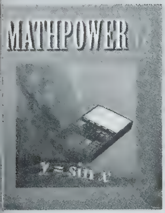
c. Because $|\sin x|$ is never negative, the only solutions would be values where x is non-negative. The roots of $|\sin x| = 0.5x$ would be roots of $\sin x = 0.5x$ excluding -1.895. (Refer to the following displays.)

$y = \sin x$ and $y = 0.5x$



$y = |\sin x|$ and $y = 0.5x$



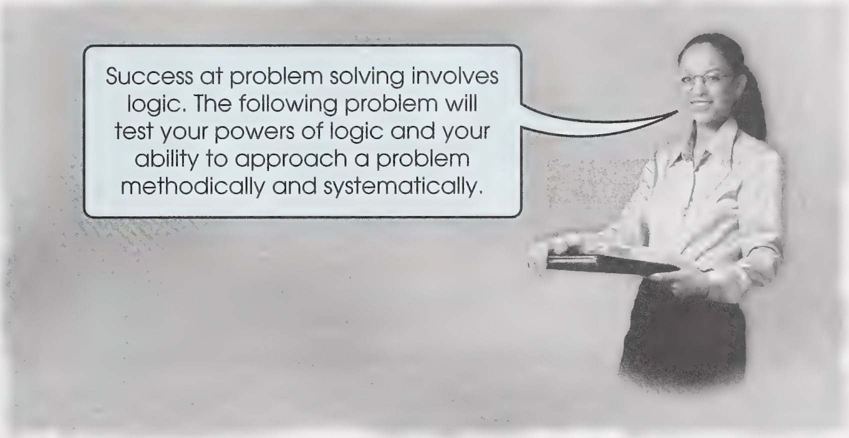


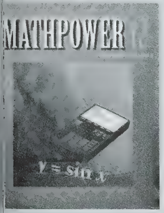
Mathematical Process
- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ☒ Problem Solving
- ☒ Reasoning
- ■ Technology
- ■ Visualization

3. Answer questions 10 and 13 of “Applications and Problem Solving” on page 257 of the textbook.

Turn to page 223 to compare your responses with those in the Appendix.

Now Try This


Success at problem solving involves logic. The following problem will test your powers of logic and your ability to approach a problem methodically and systematically.



Mathematical Process
- ☒ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ■ Technology
- ■ Visualization

4. Turn to page 257 of *MATHPOWER™ 12* and answer “LOGIC POWER.”

Turn to page 224 to compare your response with the one in the Appendix.

Looking Back

In this activity, you used technology as an aid to solving complex trigonometric equations. You discovered that adding graphing technology to your problem-solving methods reduces the time spent determining solutions and enables you to determine solutions that would be difficult or impossible algebraically.

In your journal, describe how the Intersect feature and the Zero feature on your graphing calculator are used to obtain answers to trigonometric equations.

Activity 4: Trigonometric Identities

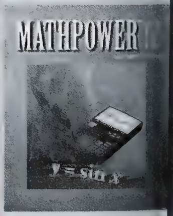
Do you know any identical twins? Often, a stranger will find it difficult to tell twins apart because of their similar appearance. However, family and friends don't have that problem. They recognize each twin's unique personality, interests, and abilities. Twins may look the same, but usually they behave quite differently.

In trigonometry, just the opposite is true! There are a number of pairs of expressions that look quite different but, in fact, are identical. They appear different, but behave the same!

In this activity, you will prove that two trigonometric expressions that appear to be different are actually identical. These statements are examples of identities.

an equation that is true for all values of the variable for which both sides of the equation are defined

Turn to page 258 of *MATHPOWER*™ *12* and read "Trigonometric Identities."



1. Answer the following questions on page 258 of the textbook:

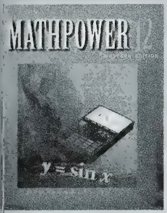
- **a.** questions a. to c. of "Explore: Use Inductive Reasoning"
- **b.** questions 1 to 5 of "Inquire"

Turn to page 225 to compare your responses with those in the Appendix.

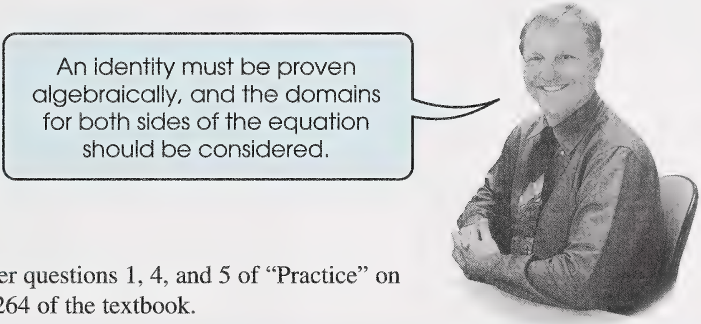
Now you will examine trigonometric identities and their proofs. You already know a number of identities—namely the definitions of the reciprocal functions: $\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$, and $\cot\theta = \frac{1}{\tan\theta}$. You will make extensive use of these identities throughout this activity.

[illegible]

n identity that
ontains at least one
igonometric function



Turn to page 259 of *MATHPOWER™ 12* and read from the top of the page to the bottom of the page, working through Example 1.


An identity must be proven algebraically, and the domains for both sides of the equation should be considered.

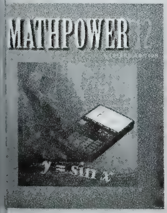
2. Answer questions 1, 4, and 5 of "Practice" on page 264 of the textbook.

Turn to page 228 to compare your responses with those in the Appendix.

Did you notice that most of the work in algebraic proofs involved using sine and cosine? It is often useful to express other functions into equivalent expressions involving sine and cosine and to manipulate these expressions algebraically.

In order to do this, you will need to know two additional identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta}$$

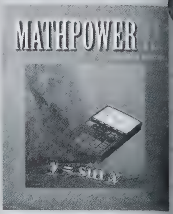


Turn to page 260 of *MATHPOWER™ 12* and read from the top of the page to Example 3, working through Example 2.

3. Answer questions 6, 8, and 10 of "Practice" on page 264 of the textbook.

Turn to page 230 to compare your responses with those in the Appendix.

Turn to pages 260 and 261 of *MATHPOWER™ 12* and work through Example 3.



Example

For $(\sin\theta - \cos\theta)^2 + (\sin\theta + \cos\theta)^2 = 2,$ do the following.

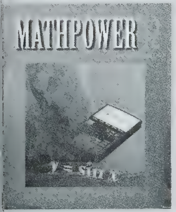
- **a.** Show that it is true for $\theta = 30°$ using exact values.
- **b.** Prove the result algebraically.
- **c.** State any restrictions.

Solution

a.
$$\begin{aligned}
\textbf{LS} &= (\sin\theta - \cos\theta)^2 + (\sin\theta + \cos\theta)^2 \\
&= (\sin 30° - \cos 30°)^2 + (\sin 30° + \cos 30°)^2 \\
&= \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)^2 \\
&= \left(\frac{1-\sqrt{3}}{2}\right)^2 + \left(\frac{1+\sqrt{3}}{2}\right)^2 \\
&= \frac{(1 - 2\sqrt{3} + 3) + (1 + 2\sqrt{3} + 3)}{4} \\
&= \frac{8}{4} \\
&= 2 \\
&= \textbf{RS}
\end{aligned}$$

b.
$$\begin{aligned}
\textbf{LS} &= (\sin\theta - \cos\theta)^2 + (\sin\theta + \cos\theta)^2 \\
&= (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta) + (\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta) \\
&= 2\sin^2\theta + 2\cos^2\theta - 2\sin\theta\cos\theta + 2\sin\theta\cos\theta \\
&= 2(\sin^2\theta + \cos^2\theta) \\
&= 2(1) \\
&= 2 \\
&= \textbf{RS}
\end{aligned}$$

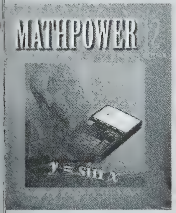
c. There are no restrictions on the domain.



4. Turn to page 264 of *MATHPOWER™ 12* and answer questions 16, 17, and 19 of "Practice."

Turn to page 233 to compare your responses with those in the Appendix.

At first, most students find proving identities difficult. To become expert in solving identities requires practice and the study of worked examples. Each question and example you explore will add invaluable insights to your problem-solving toolkit. As a result, each additional question you attack will seem easier than the ones you solved before!



Turn to pages 262 to 264 of *MATHPOWER™ 12* and work through Examples 4 to 6.

5. Answer questions 22, 23, 24, 27, 29, and 32 of "Practice" on page 264 of the textbook.

Turn to page 235 to compare your responses with those in the Appendix.

Mathematical Process
- ☐ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☒ Technology
- ☐ Visualization

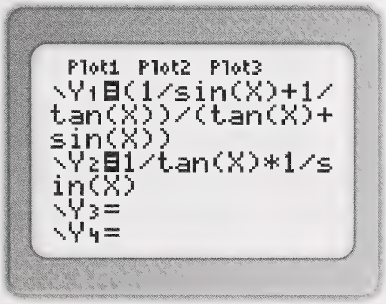
Example

For $\frac{\csc B+\cot B}{\tan B+\sin B}=\cot B\csc B$, do the following.

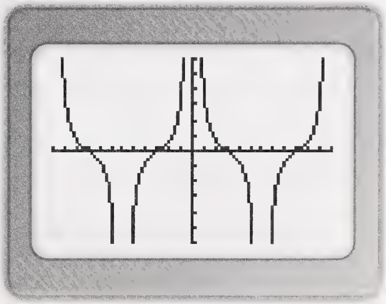
a. Use a graphing calculator to show that the identity appears to be true.
b. Prove the identity algebraically.
c. State any restrictions.

Solution

a.



```
 Plot1 Plot2 Plot3
\Y1■(1/sin(X)+1/
tan(X))/(tan(X)+
sin(X))
\Y2■1/tan(X)*1/s
in(X)
\Y3=
\Y4=
```



The graphs of the two functions appear to be identical in the viewing window.

b.
$$
\begin{aligned}
\mathbf{LS} &= \frac{\csc B + \cot B}{\tan B + \sin B} \\
&= \frac{\frac{1}{\sin B} + \frac{\cos B}{\sin B}}{\frac{\sin B}{\cos B} + \sin B} \quad \leftarrow \text{LS in terms of sine and cosine.} \\
&= \frac{\frac{1+\cos B}{\sin B}}{\frac{\sin B + \sin B \cos B}{\cos B}} \\
&= \frac{\cancel{1+\cos B}}{\sin B} \times \frac{\cos B}{\sin B\left(\cancel{1+\cos B}\right)} \\
&= \frac{1}{\sin B} \times \frac{\cos B}{\sin B} \\
&= \csc B \times \cot B \\
&= \cot B \csc B \\
&= \mathbf{RS}
\end{aligned}
$$

c. The left side has the restrictions $\sin B \neq 0$ and $\cos B \neq 0$. So, $B \neq n\pi$ and $B \neq \frac{\pi}{2} + n\pi$, where n is any integer.

The right side has the restriction $\sin B \neq 0$, since $\cot B = \frac{\cos B}{\sin B}$ and $\csc B = \frac{1}{\sin B}$. So, $B \neq n\pi$, where n is any integer.

Therefore, $\frac{\csc B + \cot B}{\tan B + \sin B} = \cot B \csc B$ is an identity with the restrictions $B \neq n\pi$ and $B \neq \frac{\pi}{2} + n\pi$, where n is any integer. These restrictions can be combined as $B \neq \frac{n\pi}{2}$, where n is any integer.

6. Turn to page 265 of *MATHPOWER*™ *12* and answer questions 36, 39, and 42 of "Applications and Problem Solving."

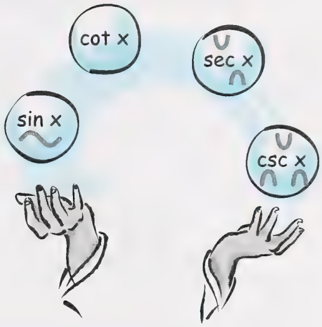
Turn to page 238 to compare your responses with those in the Appendix.

Now Try This

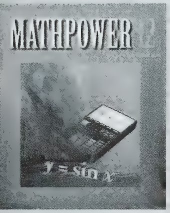
Mathematical Process

■ Communication
☒ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

When working with trigonometric identities, it often seems like you are juggling half a dozen or more things at the same time. Will a Pythagorean identity help? Is changing to a common denominator useful? Can everything be changed to sines and cosines?



Here is a little problem where you only have to juggle five things at once.



7. Turn to page 265 of *MATHPOWER*™ *12* and answer “WORD POWER.”

Turn to page 241 to compare your response with the one in the Appendix.

Looking Back

In this activity, you were introduced to trigonometric identities. You discovered that to determine if an equation is an identity, it requires an algebraic proof. You also discovered that there are values of the variable for which the identities are not true. These are the values for which the expressions in the original identity are undefined. You also discovered how useful a graphing tool, like a graphing calculator, can be when checking whether an equation was a possible identity.

Mathematical Process

☒ Communication
■ Connection
■ Estimation
■ Mental Math
■ Problem Solving
■ Reasoning
■ Technology
■ Visualization

In your journal, write the steps you would follow for determining if a trigonometric equation is a trigonometric identity. Be sure to include the steps for determining all of the excluded values in the domain. Compare your steps with those of another student taking Pure Mathematics 30.

Activity 5: Sum, Difference, and Double-Angle Identities

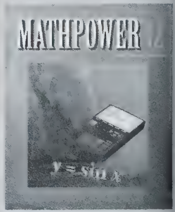
Mathematical Process

☐ Communication
☒ Connection
☐ Estimation
☐ Mental Math
☐ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

Riding a horse on a carousel is a joy for most youngsters. They move swiftly around and around and up and down all at the same time. This beautiful yet simple ride is a playground of hidden trigonometry. The inside horses move slower than the outside horses, but go around just as often during a ride. The horses also rise and fall in a regular pattern as they gallop around the track. What kind of calculations did the designer have to make to have all of this come out right? There would be radii, accelerations, tangents, gravity, masses, and motors to consider. Yet, like most modern technological marvels, the math, though very impressive, is hidden away from most people's eyes and minds.

NIK WHEELER/CORBIS

One of the tools of an engineer or a designer is the ability to solve complex problems involving trigonometric expressions. The simpler the expressions, the easier the analysis becomes. In this activity, you will explore a set of identities that will help you simplify complex expressions.



1. Turn to "INVESTIGATING MATH" on pages 266 and 267 of *MATHPOWER™ 12* and answer the following:

   a. questions 1 and 2 of Investigation 1, "Investigating $\cos(A-B)$"

   b. question 1 of Investigation 2, "Deriving an Identity for $\cos(A+B)$"

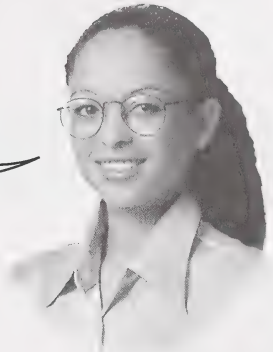
   c. questions 1 to 4 of Investigation 3, "Deriving Identities for $\sin(A+B)$, $\sin(A-B)$, $\tan(A+B)$, and $\tan(A-B)$"

   d. question 1 of Investigation 4, "Double-Angle Identities"

Turn to page 242 to compare your responses with those in the Appendix.

The work you carried out in the preceding questions provides you with some of the tools you will need to work through this activity. The quicker you can recognize identities, the quicker you will solve most of the problems involving trigonometric equations.

Turn to page 268 of *MATHPOWER™ 12* and read "Using Sum, Difference, and Double-Angle Identities"; then read from the red line near the bottom of the page to Example 1 on page 269.



You will find that recognizing these identities from both the right side and the left side is useful. In fact, you are more likely to see the expanded form (the right side) when solving problems.

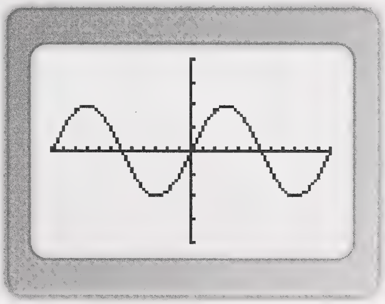
Turn to page 269 of *MATHPOWER™ 12* and work through Example 1.

2. Answer questions 1, 3, and 6 of "Practice" on page 272 of the textbook.

Turn to page 246 to compare your responses with those in the Appendix.

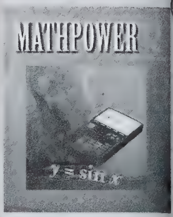
If you are confronted with a difficult identity to prove and both the left and right sides involve complicated trigonometric expressions, often you can use your graphing calculator to suggest how each side might be simplified.

For example, suppose you were trying to simplify $\tan x \cos x$. The graph of this function, suggests that it may be represented as $\sin x$.



The following examples and questions will make use of this technique.

Turn to page 269 of *MATHPOWER™ 12* and read from the red line to the bottom of page 271, working through Examples 2 and 3.

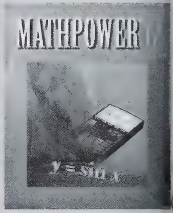


3. Answer questions 9, 12, 13, 16, 17, 21, 24, 28, 30, 32, 33, 35, and 36 of "Practice" on pages 272 and 273 of the textbook.

Turn to page 247 to compare your responses with those in the Appendix.

In the next example, you will apply your skills to a practical problem.

Turn to page 272 of *MATHPOWER™ 12* and work through Example 4.



4. Answer questions 37, 39, 42, 45, 50, 52, and 53 of "Applications and Problem Solving," on pages 273 and 274 of the textbook.

Turn to page 250 to compare your responses with those in the Appendix.

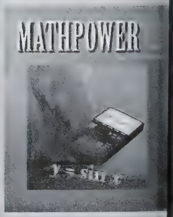
Now Try This

Patterns, again, have played a major role in this activity. Each identity is a new pattern to be looked for in problems. Now, you should be quick to recognize that the expression $\sin A \cos B - \cos A \sin B$ is the sine of a difference of angles; but will you be as quick to recognize this pattern or other patterns within a word problem?

Mathematical Process

☐ Communication
☐ Connection
☐ Estimation
☐ Mental Math
☒ Problem Solving
☐ Reasoning
☐ Technology
☐ Visualization

5. Turn to "PROBLEM SOLVING" on page 281 of *MATHPOWER™ 12* and answer question 5 of "Using the Strategies."

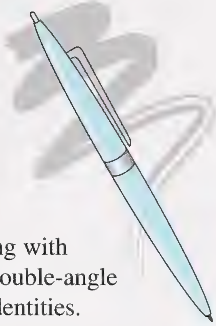


Turn to page 256 to compare your response with the one in the Appendix.

Looking Back

Mathematical Process

- ☒ Communication
- ☐ Connection
- ☐ Estimation
- ☐ Mental Math
- ☐ Problem Solving
- ☐ Reasoning
- ☐ Technology
- ☐ Visualization



In this activity, you worked with a number of new trigonometric identities. You derived and proved the sum, difference, and double-angle identities for sine, cosine, and tangent. You have seen that graphs can help you decide how to simplify expressions in an identity. You also simplified complex trigonometric equations by using trigonometric identities and applied your new knowledge of sum and difference identities to problems.

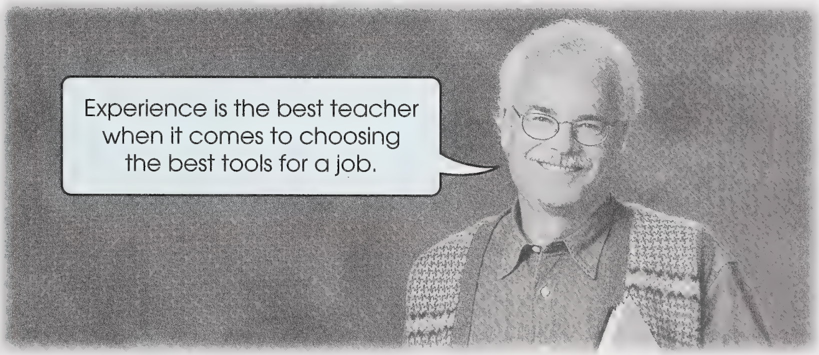
In your journal, write about the problems you encountered in working with the sum and difference identities. Write an explanation of how the double-angle identities are derived from the sum identities and the Pythagorean identities.

Follow-up Activities

If you had difficulties understanding the concepts and skills in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts and skills, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

A major part of the work when solving trigonometric equations and proving trigonometric identities is choosing the correct tool. These tools span a wide range: from reciprocal definitions, to the values of special angles, to the Pythagorean identities, to the sum and difference formulas, and to the double-angle formulas, just to name a few.


Experience is the best teacher when it comes to choosing the best tools for a job.

Example

What tools would help you simplify each of the following expressions.

a. $\sin x \csc x$

b. $\sec x \cot x$

c. $\sin x \cos x$

Solution

a. The expression $\sin x \csc x$ can be simplified by using the definition of cosecant.

$$\begin{aligned}\sin \csc x &= \sin x\left(\frac{1}{\sin x}\right)\\ &= 1\end{aligned}$$

b. The expression $\sec x \cot x$ can be simplified by using the definition of secant and the quotient identity for cotangent.

$$\begin{aligned}\sec x \cot x &= \frac{1}{\cos x}\left(\frac{\cos x}{\sin x}\right)\\ &= \frac{1}{\sin x}\\ &= \csc x\end{aligned}$$

c. The expression $\sin x \cos x$ can be simplified by using the double-angle identity for sine.

$$\begin{aligned}\sin x \cos x &= \frac{1}{2}(2 \sin x \cos x)\\ &= \frac{1}{2}\sin 2x\end{aligned}$$

1. From the following list of identities, choose the identity that will best assist you in simplifying each expression:

- $\sin^2 x + \cos^2 x = 1$
- $\sin 2x = 2 \sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $1 + \tan^2 x = \sec^2 x$
- $\sin(A+B) = \sin A \cos B + \cos A \sin B$
- $\cos(A+B) = \cos A \cos B - \sin A \sin B$

a. $\sin^2 x - 1$

b. $10 \cos^2 x - 10$

c. $\dfrac{\sin 2x}{2 \sin x}$

d. $\cos^2 x - \cos 2x$

e. $\sin^2 \dfrac{\pi}{7} \sec^2 \dfrac{\pi}{7} + 1$

f. $\sin \dfrac{7\pi}{12}$

g. $\cos 75°$

h. $1 - \sec^2 x$

2. For each expression in question 1, write a simplified expression that contains a single trigonometric function or an exact value.

Turn to page 257 to compare your responses with those in the Appendix.

When confronted with the task of solving a trigonometric equation, you will use either your calculator to find an approximate solution or your knowledge of the trigonometric values of special angles to find an exact solution.

Mathematical Process
- ■ Communication
- ■ Connection
- ■ Estimation
- ■ Mental Math
- ■ Problem Solving
- ■ Reasoning
- ☒ Technology
- ☒ Visualization

Example

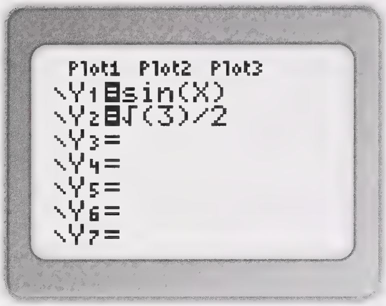
Use the graph of $y = \sin x$, where $0 \le x \le 2\pi$, to help find the exact solutions to the following equations.

a. $\sin \dfrac{x}{2} = \dfrac{\sqrt{3}}{2}$

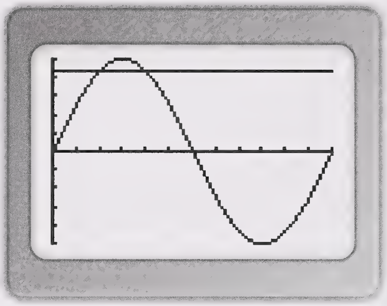
b. $\sin \dfrac{x}{3} = \dfrac{\sqrt{3}}{2}$

Solution

a. Graph $y = \sin x$ as Y_1 and $y = \frac{\sqrt{3}}{2}$ as Y_2.



```
Plot1 Plot2 Plot3
\Y1=sin(X)
\Y2=√(3)/2
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



The two graphs intersect at two locations: $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$.

Because the graph of $y = \sin \frac{x}{2}$ is a horizontal expansion of the graph of $y = \sin x$ by a factor of 2, the solutions are $x = \frac{\pi}{3} \times 2$ or $\frac{2\pi}{3}$ and $x = \frac{2\pi}{3} \times 2$ or $\frac{4\pi}{3}$.

b. Because the graph of $y = \sin \frac{x}{3}$ is a horizontal expansion of the graph of $y = \sin x$ of factor 3, the solutions are $x = \frac{\pi}{3} \times 3$ or π and $x = \frac{2\pi}{3} \times 3$ or 2π.

To review the details involved in solving trigonometric equations, open the Computer Assisted Instruction folder from the Pure Mathematics 30 Companion CD and work through the examples under *Solving Trigonometric Equations*.

3. Turn to page 278 of *MATHPOWER™ 12* and answer questions 7, 12, 20, 24.a., 24.b., and 25 of “Review.”

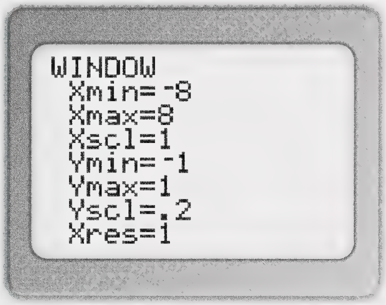
Turn to page 259 to compare your responses with those in the Appendix.

Enrichment

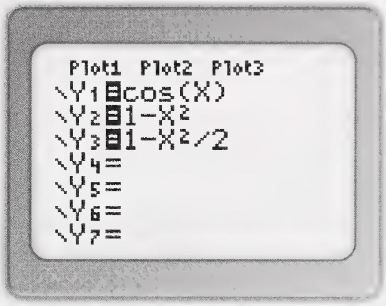
Depending on the graphing window of your graphing calculator, the graph of a polynomial function may mimic the graph of a trigonometric function. As a matter of fact, you can approximate a trigonometric function with a polynomial function.

It is easy to show that $1-x^2$ is an approximation of $\cos x$ as long as x is close to 0. This approximation becomes less and less valid as the value of $|x|$ increases.

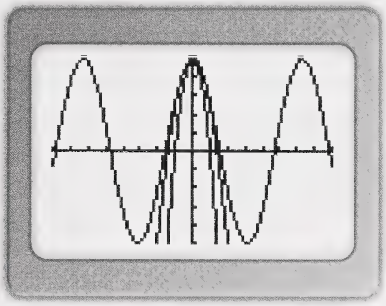
Which approximation of $\cos x$ is better, $\left(1-x^2\right)$ or $\left(1-\frac{x^2}{2}\right)$?



```
WINDOW
  Xmin=-8
  Xmax=8
  Xscl=1
  Ymin=-1
  Ymax=1
  Yscl=.2
  Xres=1
```



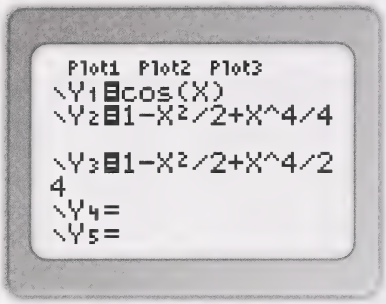
```
  Plot1  Plot2  Plot3
\Y1=cos(X)
\Y2=1-X²
\Y3=1-X²/2
\Y4=
\Y5=
\Y6=
\Y7=
```



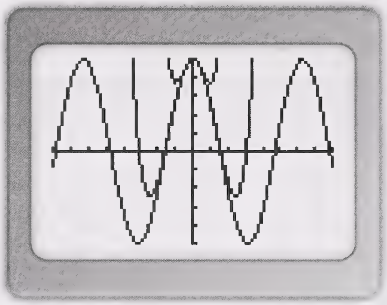
The graph shows that $\left(1-\frac{x^2}{2}\right)$ is the better approximation since it matches over a larger domain.

Because you know that cosine is an even function $\left(f(-n)=f(n)\right)$, it is reasonable to conjecture that you can approximate the cosine function with a polynomial whose terms are all even powers of x.

Which would be a better approximation of $\cos x$, $\left(1-\frac{x^2}{2}+\frac{x^4}{4}\right)$ or $\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)$?



```
Plot1 Plot2 Plot3
\Y1=cos(X)
\Y2=1-X²/2+X^4/4
\Y3=1-X²/2+X^4/2
4
\Y4=
\Y5=
```



The graph shows that $\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)$ is the better approximation.

1. Using a polynomial with powers less than 9, find a good approximation of $\cos x$ in the interval $-\pi \le x \le \pi$. (**Hint:** $2=1\times 2$ and $24=1\times 2\times 3\times 4$).

2. Using a polynomial with powers less than 8, find a good approximation of $\sin x$ in the interval $-\pi \le x \le \pi$.

Turn to page 263 to compare your responses with those in the Appendix.

From the Pure Mathematics 30 Companion CD, open the file titled *coses.xls* from the Module_Work folder. This spreadsheet allows you to experiment with finding approximations for the cosine curve using polynomials of the type you have been examining. It lets you choose the highest exponent (by entering that number in cell C2) and the range to be studied (by entering a degree value in cell D2). This spreadsheet chooses a symmetrical domain from the value you enter.

The spreadsheet graphs the cosine function in blue and the polynomial function in pink. If you can see much of the blue graph, you know you do not have an excellent fit.

Once you have chosen your values for cells C2 and D2 and have entered them, click on the "Check the fit" button to let the spreadsheet calculate and graph your results.

3. Use the file *coses.xls* to see how large of an exponent it takes to fit cosine reasonably well over the domain $-720° \le x \le 720°$.

Turn to page 263 to compare your responses with those in the Appendix.

Conclusion

In this section, you explored trigonometric equations. You investigated the relationship between the solutions of a trigonometric equation and the graph of the corresponding trigonometric function. You built on this relationship to solve trigonometric equations using pencil-and-paper techniques and technology.

You then explored a special class of equations, called identities. You discovered that you can use trigonometric identities to transform complicated expressions into simpler forms. These identities included Pythagorean, reciprocal, quotient, sum, difference, and double-angle forms. In advanced mathematics and physics courses, you will apply these identities in a variety of real-world contexts.

Solving trigonometric equations arising from models of periodic events is an important skill for mathematicians, scientists, engineers, and navigators alike. In the time of wind and sail, a ship's navigator used charts, tables, and trigonometry to locate the ship's position and to plot courses across the seas to open new lands for exploration, commerce, and settlement. Part of the romance of this period is typified by the tall ships and sailboats, some of which still operate today—a testimony to a rich history Canadians are heir to. Every time you pick up a dime, you are reminded of the Bluenose—the ship that captured the International Fishermen's Trophy in 1921 and did not relinquish it in its eighteen-year racing career. As well as the dime, the Bluenose also appears on a 2000 twenty-dollar commemorative coin in honour of its place in Canada's maritime history.

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[1]

Assignment

Turn to Assignment Booklet 5B and complete the assignment for Section 2.

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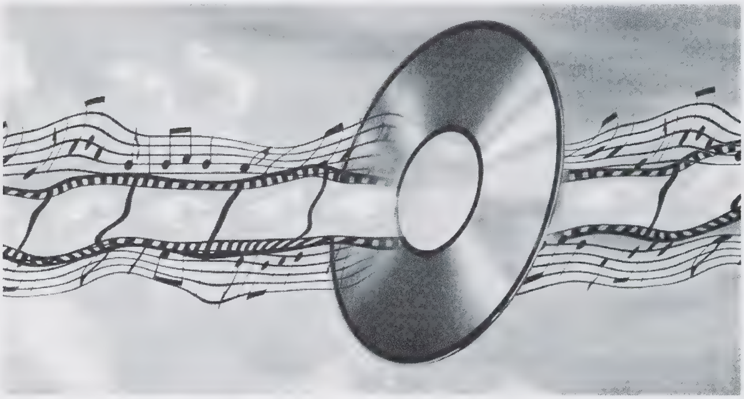
[1] This coin's photo, provided by the Royal Canadian Mint, is one in the 2000 series of sterling silver commemorative coins titled, "Land, Sea, and Rail."

Module Summary

As you prepare for the module assignment, reflect for awhile on the important ideas that you studied in this module. You explored radian measure and reviewed the definitions of the trigonometric functions and investigated their graphs. You then explored methods of solving trigonometric equations and proved a variety of trigonometric identities. All of these skills will be important as you continue your study of mathematics.



DAVID C./CORBIS

This module introduced you to one of the most interesting and historically rich fields of mathematics. The basic sine, cosine, and tangent functions have been used for millennia, helping solve practical, everyday problems. Mathematicians, technologists, and scientists are now building upon this foundation as society moves in a more technological direction. Think about the changes in the music and movie industries brought about by digital storage and compression. Hiding in the background are ideas built on trigonometry.

Final Module Assignment

Turn to Assignment Booklet 5B and complete the final module assignment. Remember to submit your completed Assignment Booklet for assessment.

Appendix

Glossary

Suggested Answers

Credits

Glossary

amplitude: half the difference between the maximum and the minimum values of a periodic function

coterminal angles: angles in standard position with the same initial arm and terminal arm

general solution: the set of all solutions in the domain of the trigonometric function appearing in the equation

identity: an equation that is true for all values of the variable for which both the left and right sides of the equation are defined

initial arm: for an angle in standard position, the arm along the positive *x*-axis

negative angle: an angle in standard position swept out by a clockwise rotation of its terminal arm

period: the smallest interval of the domain over which a periodic function repeats its values

periodic function: a function that repeats its values over a particular interval of its domain

phase shift: a horizontal translation of a periodic function

positive angle: an angle in standard position swept out by a counterclockwise rotation of its terminal arm

principal angle: the smallest positive angle coterminal with an angle in standard position

radian: a unit of angular measure determined by the size of a circle's central angle subtended by an arc the same length as the radius

reference angle: the acute angle formed by the terminal arm of an angle in standard position and the *x*-axis

sinusoidal: shaped like a sine curve

standard position: the location of an angle in the plane in which the vertex is at the origin, the initial arm lies along the positive *x*-axis, and the terminal arm is free to rotate

terminal arm: for an angle in standard position, the arm that is free to rotate

trigonometric identity: an identity that contains at least one trigonometric function

Suggested Answers

Section 1: Activity 1

1. a. **Textbook questions a. and b. of "Explore: Use a Model to Visualize," p. 186**

    a. About 6 arcs of length *r* fit around the circumference.

    b. The sector angle of one sector with arc length *r* is about 57°.

b. Textbook questions 1 to 3 of "Inquire," p. 186

1. $C = 2\pi r$

2. Because each arc length is equal to the radius, it takes 2π arc lengths to fit around the circumference exactly.

3. Because each arc length is equal to the radius, it takes π arc lengths, to fit around a semicircle. The number does not vary if the radius changes because the circumference changes by the same factor the radius changes.

2. Textbook questions 1, 2, 4, 5, 6, 10, 11, 12, 13, 16, 17, and 20 of "Practice," pp. 189 and 190

1.

degrees	0	30	45	60	90	120	135	150	180
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

$$30^\circ = \overset{1}{\cancel{30}}\left(\frac{\pi}{\underset{6}{\cancel{180}}}\text{ rad}\right)$$
$$= \frac{\pi}{6}\text{ rad}$$

$$45^\circ = \overset{1}{\cancel{45}}\left(\frac{\pi}{\underset{4}{\cancel{180}}}\text{ rad}\right)$$
$$= \frac{\pi}{4}\text{ rad}$$

$$60^\circ = \overset{1}{\cancel{60}}\left(\frac{\pi}{\underset{3}{\cancel{180}}}\text{ rad}\right)$$
$$= \frac{\pi}{3}\text{ rad}$$

2.

degrees	210	225	240	270	300	315	330
radians	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$

$$\frac{7\pi}{6}\text{ rad} = \frac{7\overset{1}{\cancel{\pi}}}{\underset{1}{\cancel{6}}}\left(\frac{\overset{30^\circ}{\cancel{180^\circ}}}{\underset{1}{\cancel{\pi}}}\right)$$
$$= 210^\circ$$

$$\frac{5\pi}{4}\text{ rad} = \frac{5\overset{1}{\cancel{\pi}}}{\underset{1}{\cancel{4}}}\left(\frac{\overset{45^\circ}{\cancel{180^\circ}}}{\underset{1}{\cancel{\pi}}}\right)$$
$$= 225^\circ$$

$$\frac{4\pi}{3}\text{ rad} = \frac{4\overset{1}{\cancel{\pi}}}{\underset{1}{\cancel{3}}}\left(\frac{\overset{60^\circ}{\cancel{180^\circ}}}{\underset{1}{\cancel{\pi}}}\right)$$
$$= 240^\circ$$

4. $0.4\text{ rad} = 0.4\left(\frac{180^\circ}{\pi}\right)$

$\doteq 22.9^\circ$

5. $-2.8\text{ rad} = -2.8\left(\frac{180^\circ}{\pi}\right)$

$\doteq -160.4^\circ$

Recall:

$\pi \doteq 3.141\,592\,654$

6. $4.25\text{ rad} = 4.25\left(\frac{180^\circ}{\pi}\right)$

$\doteq 243.5^\circ$

10. $3.14\text{ rad} = 3.14\left(\frac{180^\circ}{\pi}\right)$

$\doteq 179.9^\circ$

Section 1: Activity 1 (continued)

11. $7.75 \text{ rad} = 7.75\left(\frac{180°}{\pi}\right)$

$\doteq 444.0°$

12. $40° = 40\left(\frac{\pi}{180} \text{ rad}\right)$

$\doteq 0.70 \text{ rad}$

13. $63° = 63\left(\frac{\pi}{180} \text{ rad}\right)$

$\doteq 1.10 \text{ rad}$

16. $-100° = -100\left(\frac{\pi}{180} \text{ rad}\right)$

$\doteq -1.75 \text{ rad}$

17. $32.8° = 32.8\left(\frac{\pi}{180} \text{ rad}\right)$

$\doteq 0.57 \text{ rad}$

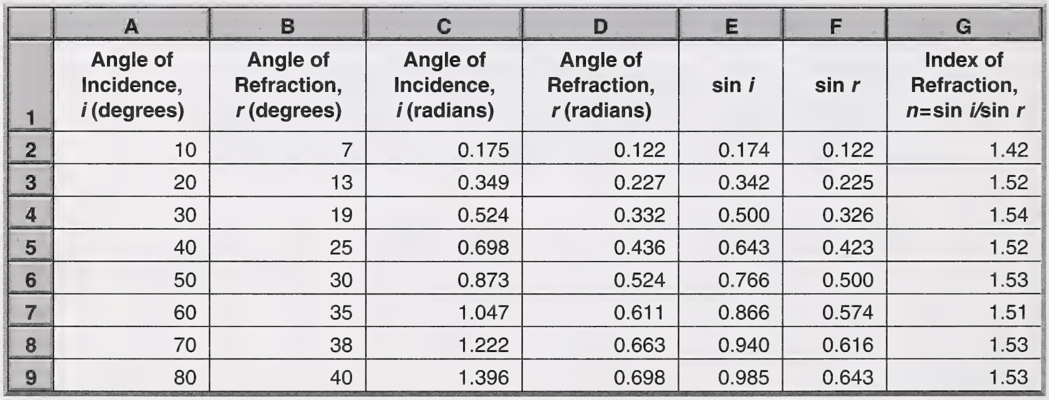
20. $-112.3° = -112.3\left(\frac{\pi}{180} \text{ rad}\right)$

$\doteq -1.96 \text{ rad}$

3. a. Textbook questions 1 and 2 of Investigation 1, “Finding the Index of Refraction of Water,” p. 192

1. The error the students made was measuring the angles of incidence and refraction in degrees and using radian mode to set up their spreadsheet. The spreadsheet interpreted each angle as an angle in radians, not degrees.

2. a. Your spreadsheet should look like the following.



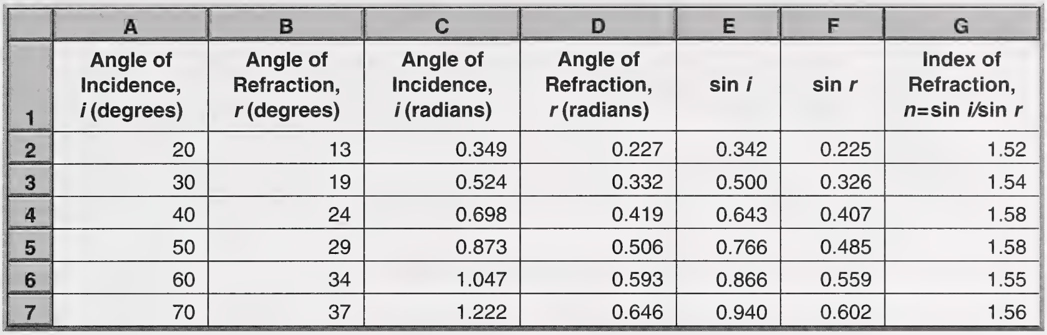
	A	B	C	D	E	F	G
1	Angle of Incidence, *i* (degrees)	Angle of Refraction, *r* (degrees)	Angle of Incidence, *i* (radians)	Angle of Refraction, *r* (radians)	sin *i*	sin *r*	Index of Refraction, *n*=sin *i*/sin *r*
2	10	7	0.175	0.122	0.174	0.122	1.42
3	20	13	0.349	0.227	0.342	0.225	1.52
4	30	19	0.524	0.332	0.500	0.326	1.54
5	40	25	0.698	0.436	0.643	0.423	1.52
6	50	30	0.873	0.524	0.766	0.500	1.53
7	60	35	1.047	0.611	0.866	0.574	1.51
8	70	38	1.222	0.663	0.940	0.616	1.53
9	80	40	1.396	0.698	0.985	0.643	1.53

b. The average index of refraction (excluding the index of 1.42) is about 1.53. The differences in the value of *n* are due to experimental error—angles were only measured to the nearest degree.

Note: The value 1.42 is extraneous. It would only skew the average.

b. Textbook questions 1 and 2 of "Investigation 2, "Determining the Index of Refraction of Other Substances," p. 193

1. **a.** Your spreadsheet should look like the following.

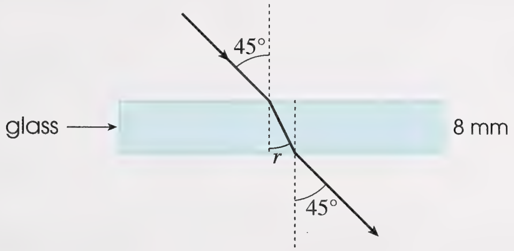


	A	B	C	D	E	F	G
1	Angle of Incidence, *i* (degrees)	Angle of Refraction, *r* (degrees)	Angle of Incidence, *i* (radians)	Angle of Refraction, *r* (radians)	sin *i*	sin *r*	Index of Refraction, *n*=sin *i*/sin *r*
2	20	13	0.349	0.227	0.342	0.225	1.52
3	30	19	0.524	0.332	0.500	0.326	1.54
4	40	24	0.698	0.419	0.643	0.407	1.58
5	50	29	0.873	0.506	0.766	0.485	1.58
6	60	34	1.047	0.593	0.866	0.559	1.55
7	70	37	1.222	0.646	0.940	0.602	1.56

The values for the index of refraction range from 1.52 to 1.58.

b. According to the spreadsheet, the average of the values for the index of refraction is approximately 1.55. When calculating each index of refraction, errors were introduced because the measures of the angle of incidence and the angle of refraction were rounded to the nearest degree. This made the value of each index either too large or too small. Calculating the average index of refraction gives a more accurate value to use than any one of the individual calculations.

c. Notice that the final ray of light is parallel to the original ray in the following sample diagram.



Section 1: Activity 1 (continued)

2. Your spreadsheet should look like the following.

	A	B	C	D	E	F	G
1	Angle of Incidence, *i* (degrees)	Angle of Refraction, *r* (degrees)	Angle of Incidence, *i* (radians)	Angle of Refraction, *r* (radians)	sin *i*	sin *r*	Index of Refraction, *n*=sin *i*/sin *r*
2	42.1	16.1	0.735	0.281	0.670	0.277	2.418
3	51.8	16.7	0.904	0.291	0.786	0.287	2.735
4	36.7	14.3	0.641	0.250	0.598	0.247	2.420

Crystal A and Crystal C are probably diamonds.

4. Textbook questions 21, 24, 26, and 27 of "Practice," p. 190

21.
$$\frac{\text{arc length}}{2\pi r}=\frac{\text{sector angle}}{2\pi}$$
$$\frac{7}{2\pi(5)}=\frac{\theta}{2\pi}$$
$$\frac{7}{5}=\theta$$
$$\theta=1.4\text{ rad}$$

24.
$$\frac{\text{arc length}}{2\pi r}=\frac{\text{sector angle}}{2\pi}$$
$$\frac{8}{2\pi(3.5)}=\frac{\theta}{2\pi}$$
$$\frac{8}{3.5}=\theta$$
$$\theta\doteq 2.3\text{ rad}$$

26.
$$\frac{\text{arc length}}{2\pi r}=\frac{\text{sector angle}}{2\pi}$$
$$\frac{x}{2\pi(5)}=\frac{1.1}{2\pi}$$
$$\frac{x}{5}=1.1$$
$$x=5.5$$

27.
$$\frac{\text{arc length}}{2\pi r}=\frac{\text{sector angle}}{2\pi}$$
$$\frac{x}{2\pi(3)}=\frac{5.6}{2\pi}$$
$$\frac{x}{3}=5.6$$
$$x=16.8$$

5. a. Textbook questions 29, 31, 35, 39, 41, 42, 45, 50, 51, 54, 59, 64, 68, and 70 of "Practice," p. 190

For questions 29, 31, 35, and 39, answers may vary. Sample answers are given.

29. positive coterminal angle $= 40^\circ + 360^\circ$
$= 400^\circ$

negative coterminal angle $= 40^\circ - 360^\circ$
$= -320^\circ$

31. positive coterminal angle $= -60^\circ + 360^\circ$
$= 300^\circ$

negative coterminal angle $= -60^\circ - 360^\circ$
$= -420^\circ$

35. positive coterminal angle $= \frac{3\pi}{4} + 2\pi$

$$= \frac{3\pi}{4} + \frac{8\pi}{4}$$

$$= \frac{11\pi}{4}$$

negative coterminal angle $= \frac{3\pi}{4} - 2\pi$

$$= \frac{3\pi}{4} - \frac{8\pi}{4}$$

$$= -\frac{5\pi}{4}$$

39. positive coterminal angle $= \frac{11\pi}{6} + 2\pi$

$$= \frac{11\pi}{6} + \frac{12\pi}{6}$$

$$= \frac{23\pi}{6}$$

negative coterminal angle $= \frac{11\pi}{6} - 2\pi$

$$= \frac{11\pi}{6} - \frac{12\pi}{6}$$

$$= -\frac{\pi}{6}$$

41. Two angles are coterminal if they differ in measure by a multiple of 2π or 360°.

$$\frac{16\pi}{3} - \frac{11\pi}{3} = \frac{5\pi}{3}$$

Therefore, the angles are not coterminal.

42. $\frac{33\pi}{8} - \frac{\pi}{8} = \frac{32\pi}{8}$

$$= 4\pi$$

$$= 2(2\pi)$$

Therefore, the angles are coterminal.

45. $371° - (-11°) = 371° + 11°$

$$= 382°$$

Therefore, the angles are not coterminal.

50. To find the principal angle for a negative angle, add multiples of 360° until a positive angle less than 360° is obtained.

$$-500° + 360° = -140°$$

$$-140° + 360° = 220°$$

The principal angle that is coterminal with $-500°$ is 220°.

Section 1: Activity 1 (continued)

51. To find the principal angle for a positive angle, subtract multiples of 360° until a positive angle less than 360° is found.

$$625^\circ - 360^\circ = 265^\circ$$

The principal angle that is coterminal with 625° is 265°.

54. To find the principal angle for a positive angle, subtract multiples of 2π until a positive value less than 2π is found.

$$\begin{aligned}\frac{11\pi}{3} - 2\pi &= \frac{11\pi}{3} - \frac{6\pi}{3} \\ &= \frac{5\pi}{3}\end{aligned}$$

The principal angle that is coterminal with $\frac{11\pi}{3}$ is $\frac{5\pi}{3}$.

59. To find the greatest negative angle coterminal with a positive angle, subtract multiples of 360° until you have a negative value that is greater than -360°.

$$305^\circ - 360^\circ = -55^\circ$$

The greatest negative angle coterminal with 305° is -55°.

64. To find the greatest negative angle coterminal with a positive angle, subtract multiples of 360° until you have a negative value that is greater than -360°.

$$700^\circ - 360^\circ = 340^\circ$$
$$340^\circ - 360^\circ = -20^\circ$$

The greatest negative angle coterminal with 700° is -20°.

68. To find the greatest negative angle coterminal with a positive angle, subtract multiples of 2π until you have a negative value that is greater than -2π.

$$\begin{aligned}\frac{17\pi}{3} - 2\pi &= \frac{17\pi}{3} - \frac{6\pi}{3} \\ &= \frac{11\pi}{3}\end{aligned} \qquad \begin{aligned}\frac{11\pi}{3} - 2\pi &= \frac{11\pi}{3} - \frac{6\pi}{3} \\ &= \frac{5\pi}{3}\end{aligned} \qquad \begin{aligned}\frac{5\pi}{3} - 2\pi &= \frac{5\pi}{3} - \frac{6\pi}{3} \\ &= -\frac{\pi}{3}\end{aligned}$$

The greatest negative angle coterminal with $\frac{17\pi}{3}$ is $-\frac{\pi}{3}$.

70. To find the greatest negative angle coterminal with a negative angle, add multiples of 2π until you have a negative value that is greater than -2π.

$$\begin{aligned}-\frac{11\pi}{2}+2\pi &= -\frac{11\pi}{2}+\frac{4\pi}{2}\\ &= -\frac{7\pi}{2}\end{aligned}$$

$$\begin{aligned}-\frac{7\pi}{2}+2\pi &= -\frac{7\pi}{2}+\frac{4\pi}{2}\\ &= -\frac{3\pi}{2}\end{aligned}$$

The greatest negative angle coterminal with $-\frac{11\pi}{2}$ is $-\frac{3\pi}{2}$.

b. Textbook questions 71, 72, 74, 75, and 77 of "Applications and Problem Solving," pp. 190 and 191

71. a.

$$\begin{aligned}\text{arc} &= 2(2\pi)\\ &= 4\pi\end{aligned}$$

$$\begin{aligned}\therefore \text{ angular speed} &= \frac{\text{arc}}{\text{time}}\\ &= \frac{4\pi}{60}\\ &\doteq 0.21\end{aligned}$$

The Ferris wheel has an average angular speed of about 0.21 rad/s.

b. If the ride lasts 5 min, a rider makes $2\times 5=10$ rotations.

The distance travelled in one rotation is the circumference of the Ferris wheel.

$$\begin{aligned}C &= 2\pi r\\ &= 2\pi(25.3)\\ &\doteq 159\text{ m}\end{aligned}$$

Therefore, after 10 rotations, a rider travels about $159\times 10=1590$ m.

72. The second hand makes one rotation $(2\pi \text{ rad})$ every 60 s.

$$\begin{aligned}\therefore \text{ angular speed} &= \frac{\text{arc}}{\text{time}}\\ &= \frac{2\pi}{60}\\ &= \frac{\pi}{30}\\ &\doteq 0.10\end{aligned}$$

The angular speed of the second hand is $\frac{\pi}{30}$ rad/s or about 0.10 rad/s.

Section 1: Activity 1 (continued)

74. The saw completes 2400 rotations every 60 s.

$$\begin{aligned}\therefore \text{ angular speed} &= \frac{\text{arc}}{\text{time}}\\ &= \frac{\overset{40}{\cancel{2400}}\times 2\pi}{\underset{1}{\cancel{60}}}\\ &= 80\pi\\ &\doteq 251.33\end{aligned}$$

The angular velocity of a point on the saw blade is 80π rad/s or about 251.33 rad/s.

75. **a.** $$\begin{aligned}2.22\times10^{4}\,{}^\circ/\text{s} &= 2.22\times10^{4}\,{}^\circ/\text{s}\times\frac{1\text{ revolution}}{360^\circ}\times\frac{60\text{ s}}{1\text{ min}}\\ &= \frac{2.22\times10^{4}\left(\overset{1}{\cancel{60}}\right)}{\underset{6}{\cancel{360}}}\\ &= 3700\end{aligned}$$

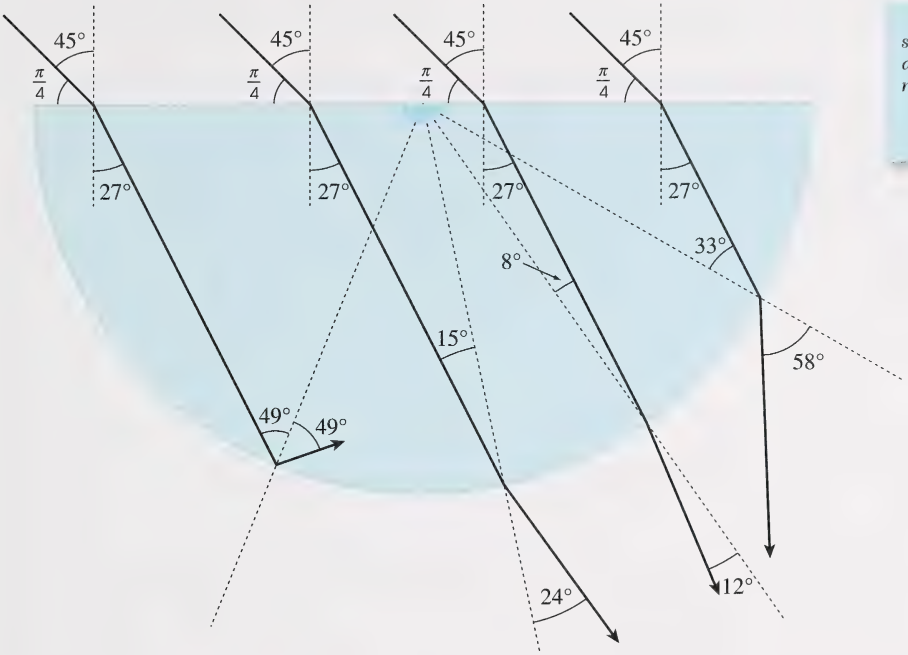
The angular speed of the pulley is 3700 rpm.

b. $$\begin{aligned}2.22\times10^{4}\,{}^\circ/\text{s} &= 2.22\times10^{4}\,{}^\circ/\text{s}\times\left(\frac{\pi}{180}\text{ rad}\right)\times\frac{60\text{ s}}{1\text{ min}}\\ &= \frac{2.22\times10^{4}\times\pi\times\overset{1}{\cancel{60}}}{\underset{3}{\cancel{180}}}\\ &= 7400\pi\end{aligned}$$

The angular speed of the pulley is 7400π rad/min.

77. Earth completes one rotation $(2\pi\text{ rad})$ each day. There are seven days in a week. Therefore, Earth rotates through $2\pi\times7=14\pi\text{ rad}$ each week.

6.



***Note:** The angles shown in the diagram are in degrees because not everyone will have a radian protractor.*

The initial rays form an angle of 45° with the normal to the surface of the hemisphere. The refracted ray inside the glass makes about a 27° angle with this same normal.

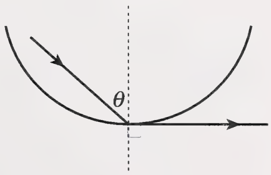
A second refraction occurs as the light ray is leaving the glass hemisphere. When the light rays exit the glass hemisphere, they are refracted away from the normal, which is along a radius of the hemisphere.

For this hemisphere, the index of refraction is

$$\begin{aligned} n &= \frac{\sin 45^\circ}{\sin 27^\circ} \\ &\doteq 1.56 \end{aligned}$$

$$\frac{\sin(\text{angle in air})}{\sin(\text{angle in glass})}$$

For an exiting ray, the largest angle it could make with the normal in the air is 90°.



Section 1: Activity 1 (continued)

In the glass, the measure of the angle between this ray and the normal is θ, where

$$\begin{aligned} 1.56 &\doteq \frac{\sin(\text{angle in air})}{\sin(\text{angle in glass})} \\ 1.56 &\doteq \frac{\sin 90^\circ}{\sin\theta} \\ 1.56 &\doteq \frac{1}{\sin\theta} \\ 1.56\sin\theta &\doteq 1 \\ \sin\theta &\doteq \frac{1}{1.56} \\ \theta &\doteq 40^\circ \end{aligned}$$

For all angles larger than 40°, the light ray is reflected internally and does not exit the glass hemisphere.

Section 1: Activity 2

1. **a. Textbook questions a. and b. of “Explore: Use a Diagram,” p. 194**

a. ΔOPC is similar to ΔOAB because their corresponding angles are equal.

$\angle AOB = \angle POC$ ← common angle

$\angle ABO = \angle PCO$ ← both right angles

$\angle OAB = \angle OPC$ ← Triangle Angle Sum Theorem

b. The ratios $\frac{OP}{OA} = \frac{PC}{AB} = \frac{OC}{OB}$ relate the sides of ΔOPC and ΔOAB.

b. Textbook questions 1 to 7 of "Inquire," p. 194

1. The length of OB is 3.

$$\begin{aligned} \therefore\ (OA)^2 &= (AB)^2 + (OB)^2 \\ (OA)^2 &= 4^2 + 3^2 \\ (OA)^2 &= 16 + 9 \\ (OA)^2 &= 25 \\ OA &= 5 \end{aligned}$$

The length of OP is 1 because it is equal to the radius of the unit circle.

2. Use the proportions that relate the sides of similar triangles, ΔOAB and ΔOPC.

$$\begin{aligned} \frac{OC}{OB} &= \frac{OP}{OA} \\ \frac{OC}{3} &= \frac{1}{5} \\ OC &= \frac{3}{5} \end{aligned}$$

3. $$\begin{aligned} \frac{PC}{AB} &= \frac{OP}{OA} \quad \longleftarrow \text{corresponding sides of similar triangles} \\ \frac{PC}{4} &= \frac{1}{5} \\ PC &= \frac{4}{5} \end{aligned}$$

4. The coordinates of P are $\left(\frac{3}{5}, \frac{4}{5}\right)$.

5. $$\begin{aligned} \sin \angle AOB &= \frac{\text{opposite}}{\text{hypotenuse}} \\ &= \frac{4}{5} \end{aligned} \qquad \begin{aligned} \cos \angle AOB &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{3}{5} \end{aligned}$$

6. The cosine ratio is the same as the x-coordinate of P, and the sine ratio is the same as the y-coordinate of P.

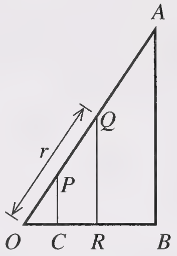
Section 1: Activity 2 (continued)

7. Points O and Q and a point, R, directly below Q on the x-axis will form another triangle similar to ΔOAB. Use these similar triangles to set up proportions to find the coordinates of Q.

$$\frac{OR}{OB} = \frac{OQ}{OA} \qquad \frac{OR}{AB} = \frac{OQ}{OA}$$

$$\frac{OR}{3} = \frac{r}{5} \qquad \frac{QR}{4} = \frac{r}{5}$$

$$OR = \frac{3r}{5} \qquad QR = \frac{4r}{5}$$



The coordinates of point Q are $\left(\frac{3r}{5}, \frac{4r}{5}\right)$.

2. **Textbook questions 1, 2, 5, 7, 9, and 12 of "Practice," p. 199**

1. The primary trigonometric ratios can be found using the values of r, x, and y. First, find r by using the Pythagorean Theorem.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{5^2 + 12^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

Now, find the six trigonometric ratios.

$$\sin\theta = \frac{y}{r} = \frac{12}{13} \qquad \cos\theta = \frac{x}{r} = \frac{5}{13} \qquad \tan\theta = \frac{y}{x} = \frac{12}{5} \text{ or } 2\frac{2}{5}$$

$$\csc\theta = \frac{r}{y} = \frac{13}{12} \qquad \sec\theta = \frac{r}{x} = \frac{13}{5} \text{ or } 2\frac{3}{5} \qquad \cot\theta = \frac{x}{y} = \frac{5}{12}$$

2. Find r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-1)^2 + (-\sqrt{3})^2} \\ &= \sqrt{1+3} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

Find the six trigonometric ratios.

$$\begin{aligned} \sin\theta &= \frac{y}{r} \\ &= \frac{-\sqrt{3}}{2} \\ &= -\frac{\sqrt{3}}{2} \end{aligned} \qquad \begin{aligned} \cos\theta &= \frac{x}{r} \\ &= \frac{-1}{2} \\ &= -\frac{1}{2} \end{aligned} \qquad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{-\sqrt{3}}{-1} \\ &= \sqrt{3} \end{aligned}$$

$$\begin{aligned} \csc\theta &= \frac{r}{y} \\ &= \frac{2}{-\sqrt{3}} \\ &= -\frac{2\sqrt{3}}{3} \end{aligned} \qquad \begin{aligned} \sec\theta &= \frac{r}{x} \\ &= \frac{2}{-1} \\ &= -2 \end{aligned} \qquad \begin{aligned} \cot\theta &= \frac{x}{y} \\ &= \frac{-1}{-\sqrt{3}} \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

5. Find r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-8)^2 + 15^2} \\ &= \sqrt{64+225} \\ &= \sqrt{289} \\ &= 17 \end{aligned}$$

$$\therefore\ \begin{aligned} \sin\theta &= \frac{y}{r} \\ &= \frac{15}{17} \end{aligned} \qquad \begin{aligned} \cos\theta &= \frac{x}{r} \\ &= \frac{-8}{17} \\ &= -\frac{8}{17} \end{aligned} \qquad \begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{15}{-8} \\ &= -\frac{15}{8} \text{ or } -1\frac{7}{8} \end{aligned}$$

Section 1: Activity 2 (continued)

7. Find r.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{(-1)^2+(-8)^2} \\ &= \sqrt{1+64} \\ &= \sqrt{65} \end{aligned}$$

$$\therefore\ \sin\theta = \frac{y}{r} = \frac{-8}{\sqrt{65}} = -\frac{8\sqrt{65}}{65} \qquad \cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{65}} = -\frac{\sqrt{65}}{65} \qquad \tan\theta = \frac{y}{x} = \frac{-8}{-1} = 8$$

9. Find r.

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ &= \sqrt{4^2+(-3)^2} \\ &= \sqrt{16+9} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$$\therefore\ \sin\theta = \frac{y}{r} = \frac{-3}{5} = -\frac{3}{5} \qquad \cos\theta = \frac{x}{r} = \frac{4}{5} \qquad \tan\theta = \frac{y}{x} = \frac{-3}{4} = -\frac{3}{4}$$

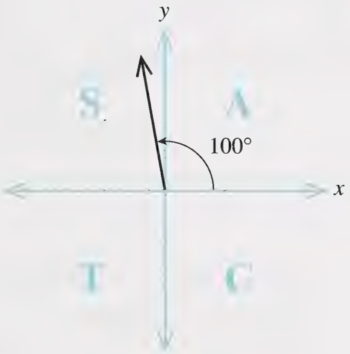
12. Find r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{6^2 + 3^2} \\ &= \sqrt{36 + 9} \\ &= \sqrt{45} \\ &= 3\sqrt{5} \end{aligned}$$

$$\therefore \ \sin\theta = \frac{y}{r} = \frac{3}{3\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \qquad \cos\theta = \frac{x}{r} = \frac{\overset{2}{6}}{\underset{1}{3\sqrt{5}}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5} \qquad \tan\theta = \frac{y}{x} = \frac{3}{6} = \frac{1}{2}$$

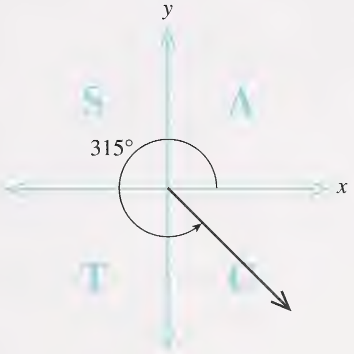
3. Textbook questions 13 to 20 of "Practice," p. 199

13.



The terminal arm is in the second quadrant. Sine is positive in this quadrant; so, $\sin 100°$ is positive.

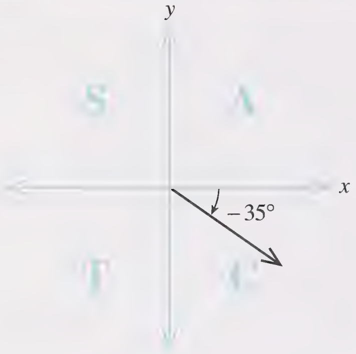
14.



The terminal arm is in the fourth quadrant. Tangent is negative in this quadrant; so, $\tan 315°$ is negative.

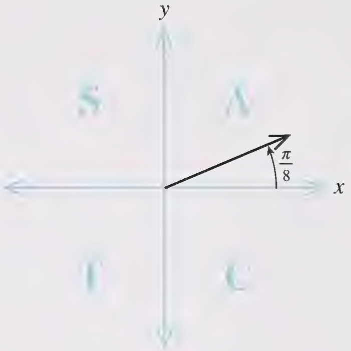
Section 1: Activity 2 (continued)

15.



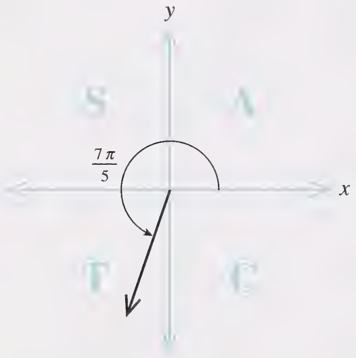
The terminal arm is in fourth quadrant. Cosine is positive in this quadrant; so, $\cos(-35°)$ is positive.

16.



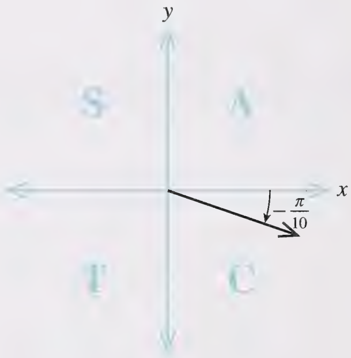
The terminal arm is in the first quadrant. Sine is positive in this quadrant; so, $\sin\frac{\pi}{8}$ is positive.

17.



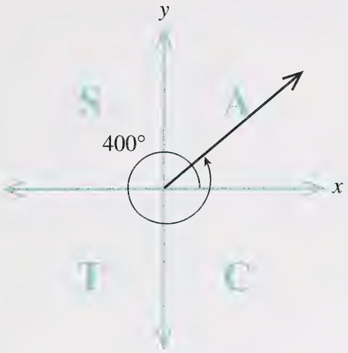
Tangent is positive in the third quadrant; so, $\cot\frac{7\pi}{5}$ is positive.

18.



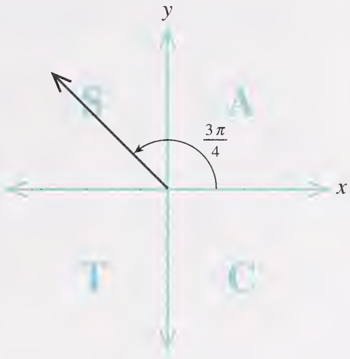
Cosine is positive in the fourth quadrant; so, $\sec\left(-\frac{\pi}{10}\right)$ is positive.

19.



Tangent is positive in the first quadrant; so, $\tan 400°$ is positive.

20.



Sine is positive in the second quadrant; so, $\sin \frac{3\pi}{4}$ is positive.

4. Textbook questions 22, 24, and 26 of "Practice," p. 200

22.
$$\begin{aligned}\sin\theta &= -\frac{1}{3}\\ &= \frac{y}{r}\end{aligned}$$

$\therefore\ y = -1$ and $r = 3$

Remember: The value of r is always positive.

Because the angle is in the fourth quadrant, x is positive.

Find x.

$$\begin{aligned}r^2 &= x^2 + y^2\\ x^2 &= r^2 - y^2\\ x &= \sqrt{r^2 - y^2}\\ &= \sqrt{3^2 - (-1)^2}\\ &= \sqrt{9-1}\\ &= \sqrt{8}\\ &= 2\sqrt{2}\end{aligned}$$

Section 1: Activity 2 (continued)

The remaining five trigonometric ratios are

$$\begin{aligned}\cos\theta &= \frac{x}{r}\\ &= \frac{2\sqrt{2}}{3}\end{aligned}$$

$$\begin{aligned}\tan\theta &= \frac{y}{x}\\ &= \frac{-1}{2\sqrt{2}}\\ &= -\frac{\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\csc\theta &= \frac{r}{y}\\ &= \frac{3}{-1}\\ &= -3\end{aligned}$$

$$\begin{aligned}\sec\theta &= \frac{r}{x}\\ &= \frac{3}{2\sqrt{2}}\\ &= \frac{3\sqrt{2}}{4}\end{aligned}$$

$$\begin{aligned}\cot\theta &= \frac{x}{y}\\ &= \frac{2\sqrt{2}}{-1}\\ &= -2\sqrt{2}\end{aligned}$$

24. $$\begin{aligned}\sec\theta &= -\frac{8}{5}\\ &= \frac{r}{x}\end{aligned}$$

$\therefore\ x = -5$ and $r = 8$

Because the angle is in the third quadrant, y is negative.

Find y.

$$\begin{aligned}y &= -\sqrt{r^2 - x^2}\\ &= -\sqrt{8^2 - (-5)^2}\\ &= -\sqrt{64 - 25}\\ &= -\sqrt{39}\end{aligned}$$

The remaining five trigonometric ratios are

$$\begin{aligned}\sin\theta &= \frac{y}{r}\\ &= \frac{-\sqrt{39}}{8}\\ &= -\frac{\sqrt{39}}{8}\end{aligned}$$

$$\begin{aligned}\cos\theta &= \frac{x}{r}\\ &= \frac{-5}{8}\\ &= -\frac{5}{8}\end{aligned}$$

$$\begin{aligned}\tan\theta &= \frac{y}{x}\\ &= \frac{-\sqrt{39}}{-5}\\ &= \frac{\sqrt{39}}{5}\end{aligned}$$

$$\begin{aligned}\csc\theta &= \frac{r}{y}\\ &= \frac{8}{-\sqrt{39}}\\ &= -\frac{8\sqrt{39}}{39}\end{aligned}$$

$$\begin{aligned}\cot\theta &= \frac{x}{y}\\ &= \frac{-5}{-\sqrt{39}}\\ &= \frac{5\sqrt{39}}{39}\end{aligned}$$

26. $\csc\theta = \frac{12}{15}$

$$= \frac{r}{y}$$

$\therefore\ y = 5$ and $r = 12$

Because the angle is in second quandrant, x is negative. Find x.

$$\begin{aligned} x &= -\sqrt{r^2 - y^2} \\ &= -\sqrt{12^2 - 5^2} \\ &= -\sqrt{144 - 25} \\ &= -\sqrt{119} \end{aligned}$$

The remaining five trigonometric ratios are

$$\begin{aligned} \sin\theta &= \frac{y}{r} \\ &= \frac{5}{12} \end{aligned}$$

$$\begin{aligned} \cos\theta &= \frac{x}{r} \\ &= \frac{-\sqrt{119}}{12} \\ &= -\frac{\sqrt{119}}{12} \end{aligned}$$

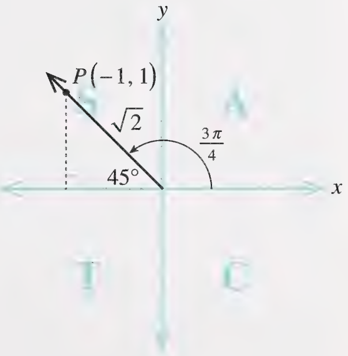
$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ &= \frac{5}{-\sqrt{119}} \\ &= -\frac{5\sqrt{119}}{119} \end{aligned}$$

$$\begin{aligned} \sec\theta &= \frac{r}{x} \\ &= \frac{12}{=-\sqrt{119}} \\ &= -\frac{12\sqrt{119}}{119} \end{aligned}$$

$$\begin{aligned} \cot\theta &= \frac{x}{y} \\ &= \frac{-\sqrt{119}}{5} \\ &= -\frac{\sqrt{119}}{5} \end{aligned}$$

5. a. Textbook questions 27, 28, 31, and 33 of "Practice," p. 200

27. Since $\pi - \frac{3\pi}{4} = \frac{\pi}{4}$, the terminal arm of the angle is in the second quadrant and the reference angle is $\frac{\pi}{4}$ or 45°.

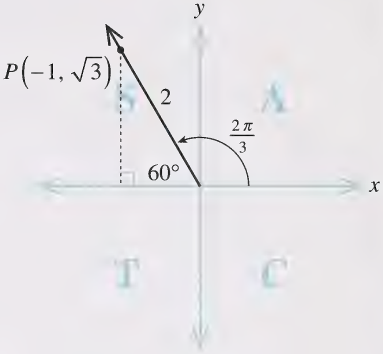


Form a 45°-45°-90° triangle (as shown). The coordinates of point P on the terminal arm are $(-1, 1)$.

$$\begin{aligned} \therefore\ \sin\frac{3\pi}{4} &= \frac{y}{r} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

Section 1: Activity 2 (continued)

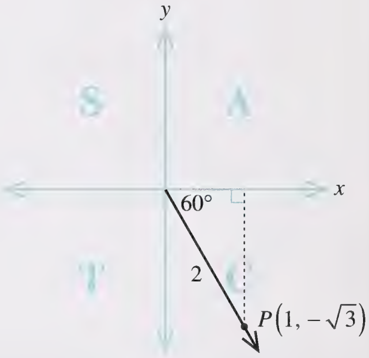
28. Since $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$, the terminal arm of the angle is in the second quadrant and the reference angle is $\frac{\pi}{3}$ or 60°.



By forming a 30°-60°-90° triangle, you see the coordinates of point P are $\left(-1, \sqrt{3}\right)$.

$$\begin{aligned}\therefore\ \tan\frac{2\pi}{3} &= \frac{y}{x}\\ &= \frac{\sqrt{3}}{-1}\\ &= -\sqrt{3}\end{aligned}$$

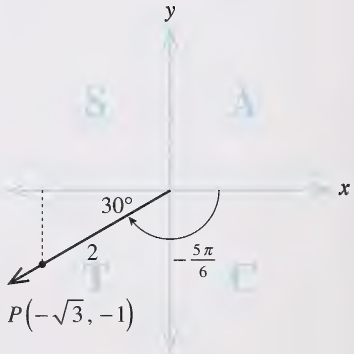
31. After a rotation of $-60°$, the terminal arm is in the fourth quadrant.



Using a 30°-60°-90° triangle, the coordinates of point P are $\left(1, -\sqrt{3}\right)$.

$$\begin{aligned}\therefore\ \sec\left(-60°\right) &= \frac{r}{x}\\ &= \frac{2}{1}\\ &= 2\end{aligned}$$

33. Since $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$, the terminal arm of the angle is in the third quadrant and the reference angle is $\frac{\pi}{6}$ or 30°.



Using a 30°-60°-90° triangle, the coordinates of point P are $\left(-\sqrt{3}, -1\right)$.

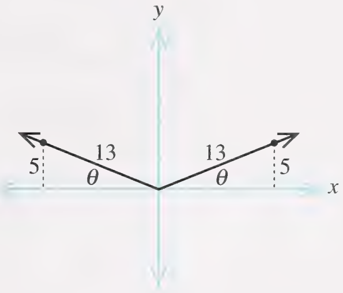
$$\begin{aligned}\therefore\ \csc\left(-\frac{5\pi}{6}\right) &= \frac{r}{y}\\ &= \frac{2}{-1}\\ &= -2\end{aligned}$$

b. Textbook questions 43, 44, 45, and 47 of "Applications and Problem Solving," p. 200

43. a. Sine is positive in the first and second quadrants.

Because $\sin\theta = \frac{5}{13}$, $y = 5$ and $r = 13$.

b. Find the possible values of x.



$$\begin{aligned} x &= \sqrt{r^2 - y^2} \\ &= \sqrt{13^2 - 5^2} \\ &= \sqrt{169 - 25} \\ &= \sqrt{144} \\ &= \pm 12 \end{aligned}$$

$$\begin{aligned} \therefore\ \cos\theta &= \frac{x}{r} \\ &= \frac{\pm 12}{13} \\ &= \pm\frac{12}{13} \end{aligned}$$

The two possible values of $\cos\theta$ are $\frac{12}{13}$ and $-\frac{12}{13}$.

44. a.

$\angle\theta$	0	90° or $\frac{\pi}{2}$	180° or π	270° or $\frac{3\pi}{2}$
$\sin\theta$	$\frac{y}{r}=\frac{0}{1}$ $=0$	$\frac{y}{r}=\frac{1}{1}$ $=1$	$\frac{y}{r}=\frac{0}{1}$ $=0$	$\frac{y}{r}=-1$ $=-1$
$\cos\theta$	$\frac{x}{r}=\frac{1}{1}$ $=1$	$\frac{x}{r}=\frac{0}{1}$ $=0$	$\frac{x}{r}=\frac{-1}{1}$ $=-1$	$\frac{x}{r}=\frac{0}{1}$ $=0$
$\tan\theta$	$\frac{y}{x}=\frac{0}{1}$ $=0$	undefined	$\frac{y}{x}=\frac{0}{-1}$ $=0$	$\frac{y}{x}=\frac{-1}{0}$ = undefined
$\csc\theta$	$\frac{r}{y}=\frac{1}{0}$ = undefined	$\frac{r}{y}=\frac{1}{1}$ $=1$	$\frac{r}{y}=\frac{1}{0}$ = undefined	$\frac{r}{y}=\frac{1}{-1}$ $=-1$
$\sec\theta$	$\frac{r}{x}=\frac{1}{1}$ $=1$	$\frac{r}{x}=\frac{1}{0}$ = undefined	$\frac{r}{x}=\frac{1}{-1}$ $=-1$	$\frac{r}{x}=\frac{1}{0}$ = undefined
$\cot\theta$	$\frac{x}{y}=\frac{1}{0}$ = undefined	$\frac{x}{y}=\frac{0}{1}$ $=0$	$\frac{x}{y}=\frac{-1}{0}$ = undefined	$\frac{x}{y}=\frac{0}{-1}$ $=0$

Section 1: Activity 2 (continued)

b. Because division by zero is undefined, $\tan 90°$ is undefined.

45. **a.** If $\tan\theta = \frac{y}{x} = 1$, then $x = y$. Thus, any point of the form (N, N), where N is a negative number, will be on the terminal arm of the angle. One example would be $(-1, -1)$.

b. Find r.

$$\begin{aligned} r^2 &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-1)^2 + (-1)^2} \\ &= \sqrt{1+1} \\ &= \sqrt{2} \end{aligned}$$

$$\therefore\ \sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad \cos\theta = \frac{x}{r} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \qquad \csc\theta = \frac{r}{y} = \frac{\sqrt{2}}{-1} = -\sqrt{2} \qquad \sec\theta = \frac{r}{x} = \frac{\sqrt{2}}{-1} = -\sqrt{2} \qquad \cot\theta = \frac{x}{r} = \frac{-1}{-1} = 1$$

47. According to the CAST rule, $\sec\theta$—the reciprocal of $\cos\theta$—is negative in the second and third quadrants.

$$\begin{aligned} \sec\theta &= -3 \\ &= -\frac{3}{1} \\ &= \frac{r}{x} \end{aligned}$$

$\therefore\ x = -1$ and $r = 3$

Remember: The value of r is always positive.

To find the tangent values, you need to find the possible values of y.

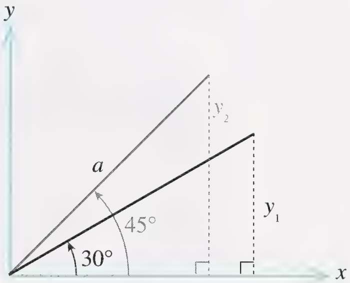
$$\begin{aligned} y &= \sqrt{r^2 - x^2} \\ &= \sqrt{3^2 - (-1)^2} \\ &= \sqrt{9-1} \\ &= \sqrt{8} \\ &= \pm 2\sqrt{2} \end{aligned}$$

$$\therefore \cos\theta = \frac{1}{\sec\theta} = \frac{1}{-3} = -\frac{1}{3}$$

$$\tan\theta = \frac{y}{x} = \frac{\pm 2\sqrt{2}}{-1} = \pm 2\sqrt{2}$$

6. Textbook questions 49, 52.a., 52.b., 52.c., and 54 of "Applications and Problem Solving," pp. 200 and 201

49. a. Sketch a diagram.



Let y_1 be the minimum height, and let y_2 be the maximum height.

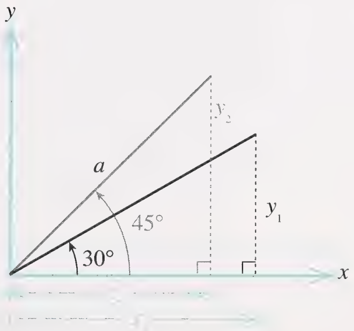
Use exact trigonometric ratios to find expressions for y_1 and y_2.

$$\sin 30° = \frac{y_1}{a} \qquad \frac{1}{2} = \frac{y_1}{a} \qquad y_1 = \frac{a}{2}$$

$$\sin 45° = \frac{y_2}{a} \qquad \frac{\sqrt{2}}{2} = \frac{y_2}{a} \qquad y_2 = \frac{a\sqrt{2}}{2}$$

The minimum height is given by $y_1 = \frac{a}{2}$, and the maximum height is given by $y_2 = \frac{a\sqrt{2}}{2}$.

Let x_1 be the maximum distance, and let x_2 be the minimum distance.



Use exact trigonometric ratios to find expressions for x_1 and x_2.

$$\cos 30° = \frac{x_1}{a} \qquad \frac{\sqrt{3}}{2} = \frac{x_1}{a} \qquad x_1 = \frac{a\sqrt{3}}{2}$$

$$\cos 45° = \frac{x_2}{a} \qquad \frac{\sqrt{2}}{2} = \frac{x_2}{a} \qquad x_2 = \frac{a\sqrt{2}}{2}$$

The minimum distance of the gangplank from the pier is $x_2 = \frac{a\sqrt{2}}{2}$, and the maximum distance of the gangplank from the pier is $x_1 = \frac{a\sqrt{3}}{2}$.

Section 1: Activity 2 (continued)

52. **a.**

LS	RS
$\dfrac{\sin\theta}{\cos\theta}$	$\tan\theta$
$=\dfrac{\frac{y}{r}}{\frac{x}{r}}$	$=\dfrac{y}{x}$
$=\dfrac{y}{x}$	

LS = RS

b. The restriction $\theta \neq \frac{\pi}{2} + k\pi$, where k is an integer, is needed because $\cos\theta$ is zero at these values. Recall that division by zero is undefined. Also, $\tan\theta$ is not defined at these values.

c. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, where $\theta \neq k\pi$, where k is an integer

Check

LS	RS
$\cot\theta$	$\dfrac{\cos\theta}{\sin\theta}$
$=\dfrac{x}{y}$	$=\dfrac{\frac{x}{r}}{\frac{y}{r}}$
	$=\dfrac{x}{y}$

LS = RS

54. **a.** If the line contains the terminal arm of an angle in standard position, the line passes through the origin making $b = 0$. The equation of the line now becomes $y = mx + 0$ or $y = mx$.

b. The slope, m, of a line is found by calculating $\frac{\text{rise}}{\text{run}}$.

$$\begin{aligned}\text{slope, } m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{y}{x} \\ &= \tan\theta\end{aligned}$$

Therefore, $\tan\theta$ is equal to the slope, m, of the equation $y = mx + b$.

7. **a.** **Textbook question 1 of Investigation 1, "The Nature of a Sine Curve," p. 203**

1. **a.** Xmin = 0

Xmax = 360

Xscl = 30

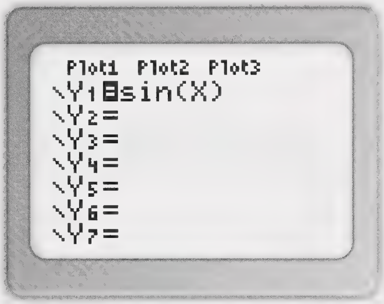
b. The maximum value of $\sin x$ is 1, and the minimum value of $\sin x$ is -1.

$\therefore$ Ymin = -1.2

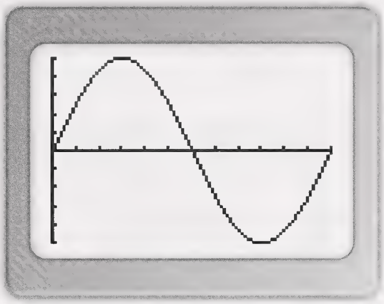
Ymax = 1.2

Yscl = 0.2

c.



```
Plot1 Plot2 Plot3
\Y1■sin(X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



The x-intercepts are at 0, 180°, and 360°.

The y-intercept is at 0.

The maximum value is 1, and it occurs at $x = 90°$.

The minimum value is -1, and it occurs at $x = 270°$.

The graph is positive when $0° < x < 180°$.

The graph is negative when $180° < x < 360°$.

According to the CAST rule, sine is positive in the first and second quadrants (0° to 180°) and negative in the third and fourth quadrants (180° to 360°). This matches what the graph shows.

Section 1: Activity 2 (continued)

b. Textbook question 1 of Investigation 2, "The Nature of a Cosine Curve," p. 203

1. a. $\text{Xmin} = 0$

$\text{Xmax} = 2\pi\left(\text{or } 6.283\ 185\ 3\ldots\right)$

$\text{Xscl} = \frac{\pi}{6}\left(\text{or } 0.523\ 598\ 77\ldots\right)$

$\frac{\pi}{6}\text{ rad} = 30°$

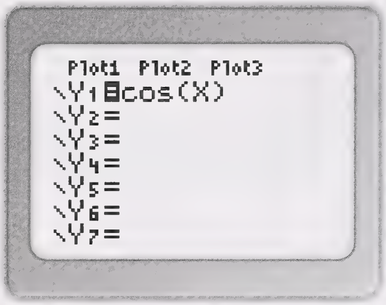
b. The maximum value of $\cos x$ is 1, and the minimum value of $\cos x$ is -1.

$\therefore\ \text{Ymin} = -1.2$

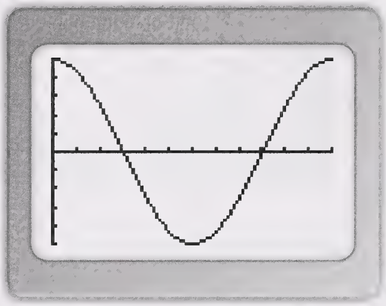
$\text{Ymax} = 1.2$

$\text{Yscl} = 0.2$

c.



```
Plot1 Plot2 Plot3
\Y1=cos(X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



The x-intercepts are at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

The y-intercept is at 1.

The maximum value is 1, and it occurs at $x = 0$ and $x = 2\pi$.

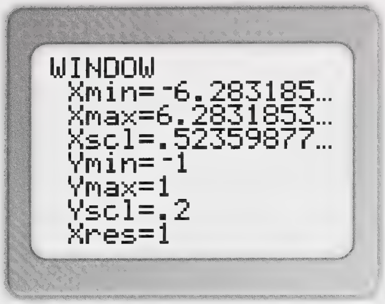
The minimum value is -1, and it occurs at $x = \pi$.

The function is positive when $0 \le x < \frac{\pi}{2}$ and when $\frac{3\pi}{2} < x \le 2\pi$.

The function is negative when $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

c. Textbook question 1 of Investigation 3, "Comparing Sine and Cosine Curves," p. 203

1. a.

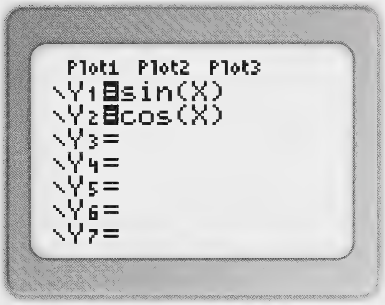


```
WINDOW
 Xmin=-6.283185…
 Xmax=6.2831853…
 Xscl=.52359877…
 Ymin=-1
 Ymax=1
 Yscl=.2
 Xres=1
```

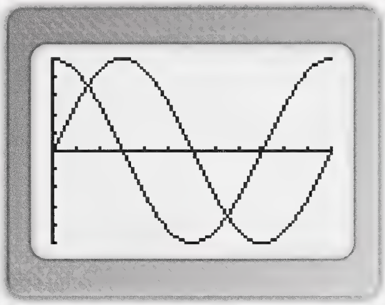
$\text{Xmin} = -2\pi$

$\text{Xmax} = 2\pi$

$\text{Xscl} = \frac{\pi}{6}$



```
Plot1 Plot2 Plot3
\Y1■sin(X)
\Y2■cos(X)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



The graphs have the same maximum and minimum values, repeat at the same frequency, and have the same shape. The graphs differ in *x*- and *y*-intercepts.

b. There is a transformation that relates the cosine and sine graphs. The cosine graph can be translated $\frac{\pi}{2}$ to the right to become the sine graph.

8. Textbook questions 35 to 42 of "Practice," p. 200

35. $\tan 240° \doteq 1.7321$

36. $\cos\left(-\frac{5\pi}{4}\right) \doteq -0.7071$

37. $\sec\frac{7\pi}{3} = 2$

Note: $\sec\frac{7\pi}{3} = \frac{1}{\cos\frac{7\pi}{3}}$

38. $\sin(-270°) = 1$

39. $\cos(-120°) = -0.5$

Section 1: Activity 2 (continued)

40. $\cos 150° \doteq -0.8660$

41. $\sin\left(-\frac{\pi}{4}\right) \doteq -0.7071$

42. $\csc 675° \doteq -1.4142$

Note: $\csc 675° = \frac{1}{\sin 675°}$

9. The sequence of numbers is actually the different representations of the number 17 in the bases 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, and 4. The next member of the sequence is 122; it is the representation of 17 in base 3.

Base	Expansion	Representation of 17 in the Given Base
16	$1(16)+1$	11
15	$1(15)+2$	12
14	$1(14)+3$	13
13	$1(13)+4$	14
12	$1(12)+5$	15
11	$1(11)+6$	16
10	$1(10)+7$	17
9	$1(9)+8$	18
8	$2(8)+1$	21
7	$2(7)+3$	23
6	$2(6)+5$	25
5	$3(5)+2$	32
4	$1(16)+0(4)+1$	101
3	$1(9)+2(3)+2$	122
2	$1(16)+0(8)+0(4)+0(2)+1$	10 001

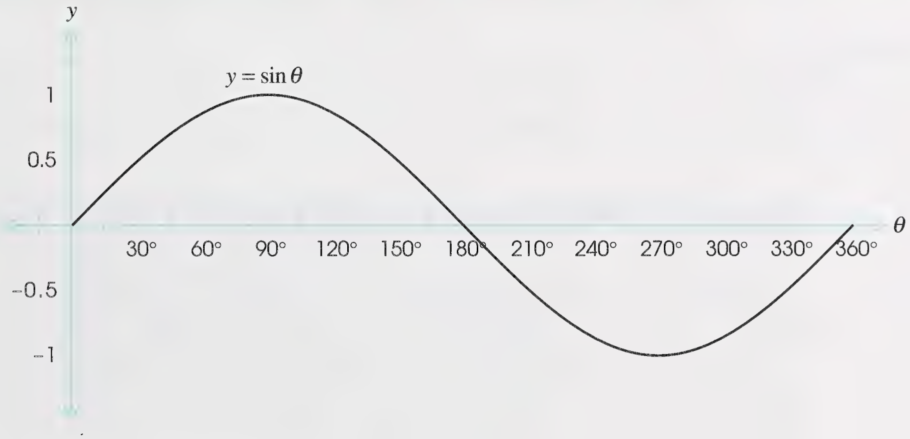
Section 1: Activity 3

1. a. Textbook questions a. and b. of "Explore: Draw a Graph," p. 204

a.

θ	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
θ	210°	225°	240°	270°	300°	315°	330°	360°	
sin θ	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	

b.



b. Textbook questions 1 to 3 of "Inquire," p. 204

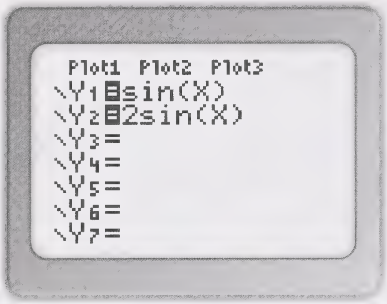
1. The maximum value of $\sin\theta$ is 1, and the minimum value of $\sin\theta$ is -1.

2. The x-intercepts are 0, 180°, and 360°.

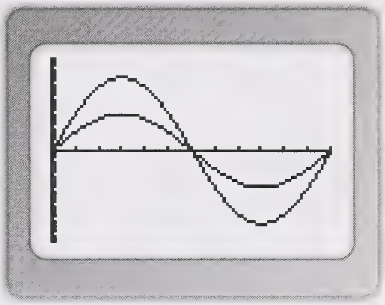
3. a. If the function were $y = 2\sin\theta$, the values for $\sin\theta$ in the table would double. Therefore, the maximum value would be 2 and the minimum value would be -2. However, the x-intercepts would remain at 0, 180°, and 360° because $2 \times 0 = 0$.

Section 1: Activity 3 (continued)

b. The graphing calculator shows the following.



```
Plot1 Plot2 Plot3
\Y1■sin(X)
\Y2■2sin(X)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



2. Textbook question 1 of "Period and Amplitude," p. 202

1. a.

$$\text{Amplitude} = \frac{\text{maximum} - \text{minimum}}{2} = \frac{4-(-4)}{2} = 4$$

$$\text{Period} = 6 - 0 = 6$$

b.

$$\text{Amplitude} = \frac{\text{maximum} - \text{minimum}}{2} = \frac{2-0}{2} = 1$$

$$\text{Period} = 7 - 3 = 4$$

c.

$$\text{Amplitude} = \frac{\text{maximum} - \text{minimum}}{2} = \frac{1.5-(-1.5)}{2} = 1.5$$

$$\text{Period} = 2 - 0 = 2$$

d.

$$\text{Amplitude} = \frac{\text{maximum} - \text{minimum}}{2} = \frac{6-(-6)}{2} = 6$$

$$\text{Period} = 8 - 0 = 8$$

3. Textbook questions 1, 4, 5, 9, and 10 of "Practice," p. 209

1. The amplitude of $y = 5 \sin \theta$ is 5.

4. The amplitude of $y = -3.5 \sin \theta$ is 3.5.

5. The amplitude of $y = -\sin \theta$ is 1.

9. The amplitude of $y = -0.4 \sin \theta$ is 0.4.

10. The amplitude of $y = 120 \cos \theta$ is 120.

4. Textbook questions 12, 13, 14, 17, and 20 of "Practice," p. 209

12. $$\text{Period} = \frac{360°}{|3|} = 120°$$

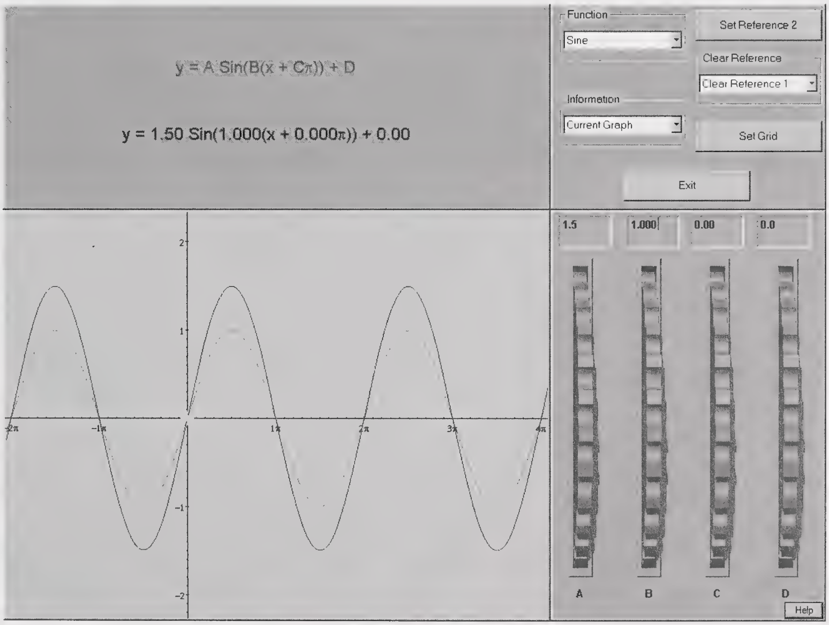
13. $$\text{Period} = \frac{360°}{\left|\frac{3}{4}\right|} = 360° \times \frac{4}{3} = 480°$$

14. $$\text{Period} = \frac{360°}{|0.5|} = 720°$$

17. $$\text{Period} = \frac{360°}{\left|\frac{1}{3}\right|} = 360° \times \frac{3}{1} = 1080°$$

20. $$\text{Period} = \frac{360°}{\left|\frac{3}{10}\right|} = 360° \times \frac{10}{3} = 1200°$$

5. a. When the value of *A* changes, the amplitude changes. If the value of *A* increases, the amplitude increases, and vice versa. For example, when the value of *A* changes from 1 to 1.5, the amplitude changes from 1 to 1.5.



y = A Sin(B(x + Cπ)) + D

y = 1.50 Sin(1.000(x + 0.000π)) + 0.00

Function: Sine

Information: Current Graph

Clear Reference: Clear Reference 1

A: 1.5

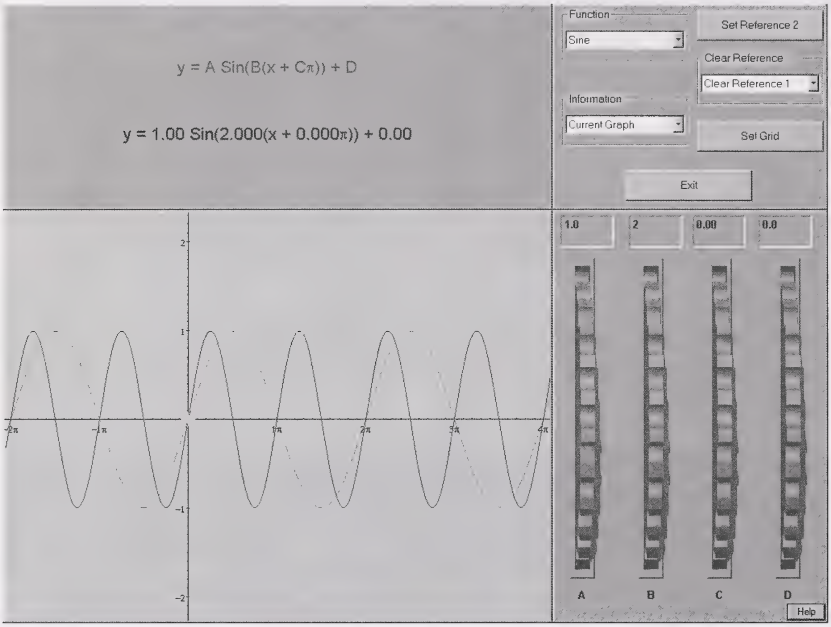
B: 1.000

C: 0.00

D: 0.0

Section 1: Activity 3 (continued)

b. When the value of B changes, the period changes. If the value of B increases, the period decreases, and vice versa. For example, when the value of B changes from 1 to 2, the period changes from 2π to π.


$y = A \operatorname{Sin}(B(x + C\pi)) + D$

$y = 1.00 \operatorname{Sin}(2.000(x + 0.000\pi)) + 0.00$

Function: Sine

Information: Current Graph

Clear Reference: Clear Reference 1

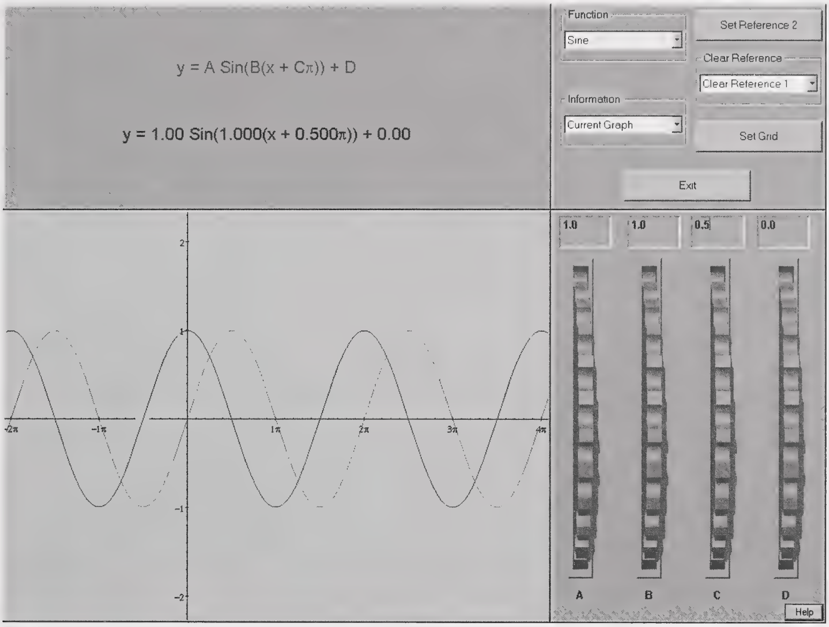
A: 1.0

B: 2

C: 0.00

D: 0.0

c. When the value of C changes, the graph either moves to the left or to the right. If $C > 0$, the graph shifts to the left the corresponding amount. If $C < 0$, the graph shifts to the right the corresponding amount. For example, when the value of C changes from 0.0 to 0.5, the graph shifts to the left.



y = A Sin(B(x + Cπ)) + D

y = 1.00 Sin(1.000(x + 0.500π)) + 0.00

Function: Sine

Information: Current Graph

Clear Reference: Clear Reference 1

A: 1.0

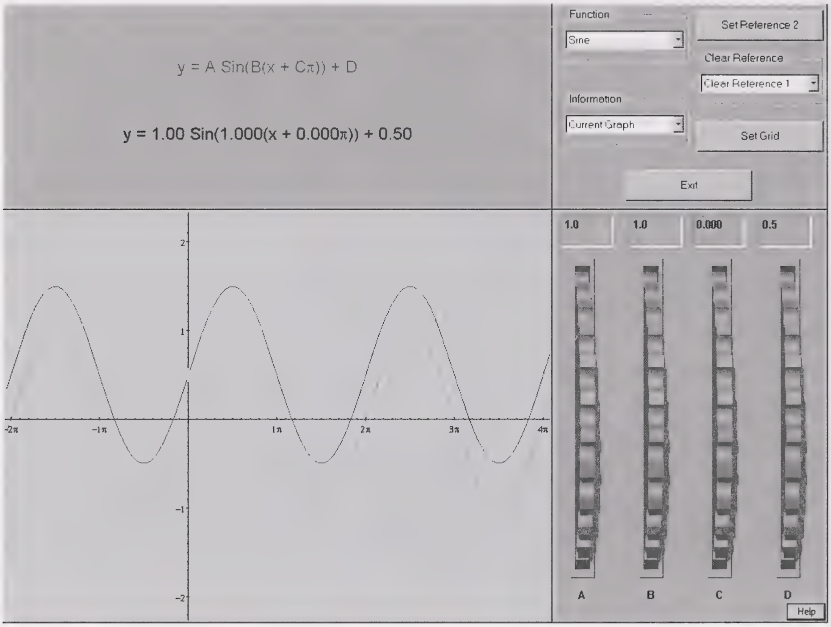
B: 1.0

C: 0.5

D: 0.0

Section 1: Activity 3 (continued)

d. When the value of D changes, the graph either moves up or down. If $D > 0$, the graph moves up the corresponding amount. If $D < 0$, the graph moves down the corresponding amount. For example, when the value of D changes from 0.0 to 0.5, the graph moves up 0.5 units.



$y = A \text{ Sin}(B(x + C\pi)) + D$

$y = 1.00 \text{ Sin}(1.000(x + 0.000\pi)) + 0.50$

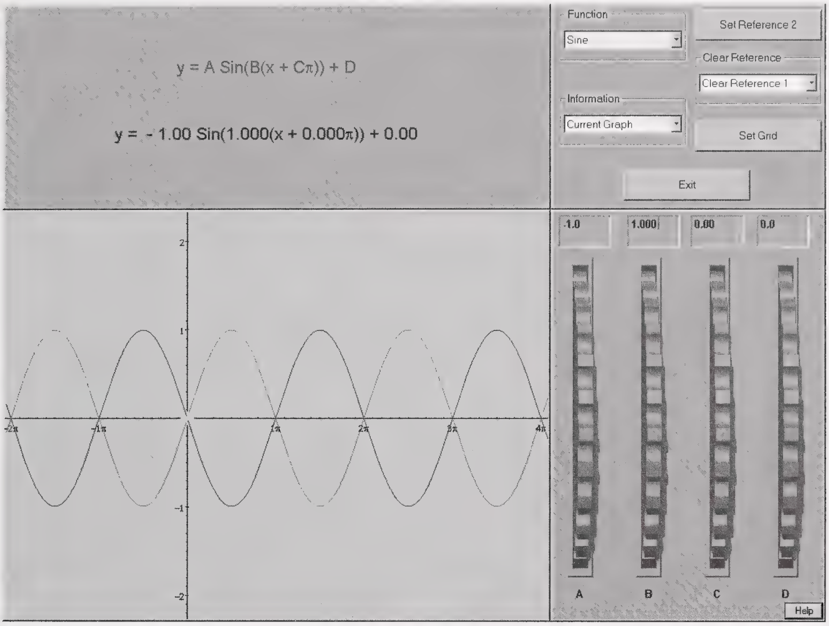
Function: Sine

Information: Current Graph

Clear Reference: Clear Reference 1

A: 1.0
B: 1.0
C: 0.000
D: 0.5

e. When the value of A changes to a negative number, the graph flips about its midline. For example, when the value of A changes from 1.0 to -1.0, the graphs become mirror images in the x-axis.



y = A Sin(B(x + Cπ)) + D

y = - 1.00 Sin(1.000(x + 0.000π)) + 0.00

Function: Sine

Information: Current Graph

Clear Reference: Clear Reference 1

A: -1.0

B: 1.000

C: 0.00

D: 0.0

Section 1: Activity 3 (continued)

6. Textbook questions 22, 23, 25, 28, and 29 of "Practice," p. 209

22. Compare $y = -3\cos 2\theta$ with $y = a\cos b\theta$.

$$\therefore \text{Amplitude} = |-3| \\ = 3$$

$$\text{Period} = \frac{2\pi}{|2|} \\ = \pi$$

23. Compare $y = 0.5\cos\frac{3\theta}{5}$ with $y = a\cos b\theta$.

$$\therefore \text{Amplitude} = |0.5| \\ = 0.5$$

$$\text{Period} = \frac{2\pi}{\left|\frac{3}{5}\right|} \\ = 2\pi \times \frac{5}{3} \\ = \frac{10\pi}{3}$$

25. Compare $y = 3\sin\frac{2\theta}{3}$ with $y = a\sin b\theta$.

$$\therefore \text{Amplitude} = |3| \\ = 3$$

$$\text{Period} = \frac{2\pi}{\left|\frac{2}{3}\right|} \\ = \cancel{2}\pi \times \frac{3}{\cancel{2}} \\ = 3\pi$$

28. Compare $y = 75\sin 12\theta$ with $y = a\sin b\theta$.

$$\therefore \text{Amplitude} = |75| \\ = 75$$

$$\text{Period} = \frac{2\pi}{|12|} \\ = \frac{\pi}{6}$$

29. Compare $y = -6.8\sin 7.2\theta$ with $y = a\sin b\theta$.

$$\therefore \text{Amplitude} = |-6.8| \\ = 6.8$$

$$\text{Period} = \frac{2\pi}{|7.2|} \\ = \frac{\pi}{3.6} \\ = \frac{5\pi}{18}$$

7. a. Textbook questions 31, 35, 36, 39, 41, 43, 45, 48, and 50 of "Practice," pp. 209 and 210

31. $$\text{Amplitude} = |a|$$
$$\frac{1}{3} = |a|$$
$$a = \pm\frac{1}{3}$$

$$\text{Period} = \frac{2\pi}{|b|}$$
$$\pi = \frac{2\pi}{|b|}$$
$$|b| = \frac{2\pi}{\pi}$$
$$|b| = 2$$
$$b = \pm 2$$

Therefore, the possible equations are $y = \frac{1}{3}\cos 2\theta$, $y = -\frac{1}{3}\cos 2\theta$, $y = \frac{1}{3}\cos(-2\theta)$, and $y = -\frac{1}{3}\cos(-2\theta)$.

35. $$\text{Amplitude} = |a|$$
$$2.8 = |a|$$
$$a = \pm 2.8$$

$$\text{Period} = \frac{360°}{|b|}$$
$$90° = \frac{360°}{|b|}$$
$$|b| = \frac{360°}{90°}$$
$$|b| = 4$$
$$b = \pm 4$$

Therefore, the possible equations are $y = 2.8\cos 4\theta$, $y = -2.8\cos 4\theta$, $y = 2.8\cos(-4\theta)$, and $y = -2.8\cos(-4\theta)$.

36. $$\text{Amplitude} = |a|$$
$$6 = |a|$$
$$a = \pm 6$$

$$\text{Period} = \frac{360°}{|b|}$$
$$180° = \frac{360°}{|b|}$$
$$|b| = \frac{360°}{180°}$$
$$|b| = 2$$
$$b = \pm 2$$

Therefore, the possible equations are $y = 6\sin 2\theta$, $y = -6\sin 2\theta$, $y = 6\sin(-2\theta)$, and $y = -6\sin(-2\theta)$.

Section 1: Activity 3 (continued)

39.
$$\begin{aligned}\text{Amplitude} &= |a| \\ 15 &= |a| \\ a &= \pm 15\end{aligned}$$

$$\begin{aligned}\text{Period} &= \frac{2\pi}{|b|} \\ \frac{\pi}{3} &= \frac{2\pi}{|b|} \\ |b| &= \frac{2\pi}{\frac{\pi}{3}} \\ |b| &= 6 \\ b &= \pm 6\end{aligned}$$

Therefore, the possible equations are $y = 15 \sin 6\theta$, $y = -15 \sin 6\theta$, $y = 15 \sin(-6\theta)$, and $y = -15 \sin(-6\theta)$.

41.
$$\begin{aligned}\text{Amplitude} &= \frac{|\text{max} - \text{min}|}{2} \\ &= \frac{|4-(-4)|}{2} \\ &= \frac{8}{2} \\ &= 4\end{aligned}$$

$$\begin{aligned}\text{Period} &= \frac{360^\circ}{|b|} \\ 360^\circ &= \frac{360^\circ}{|b|} \\ |b| &= \frac{360^\circ}{360^\circ} \\ b &= 1\end{aligned}$$

The equation is $y = 4 \sin x$.

43.
$$\begin{aligned}\text{Amplitude} &= \frac{|\text{max} - \text{min}|}{2} \\ &= \frac{|3-(-3)|}{2} \\ &= \frac{6}{2} \\ &= 3\end{aligned}$$

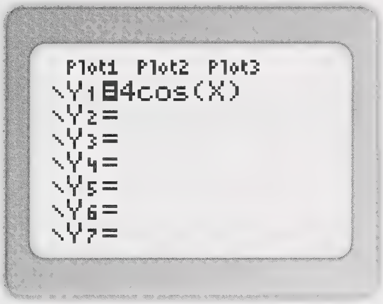
$$\begin{aligned}\text{Period} &= \frac{2\pi}{|b|} \\ 8\pi &= \frac{2\pi}{|b|} \\ |b| &= \frac{2\pi}{8\pi} \\ b &= \frac{1}{4}\end{aligned}$$

The equation is $y = 3 \cos \frac{x}{4}$.

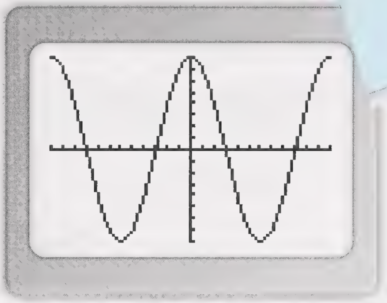
45. Compare $y=-4\cos x$ with $y=a\cos bx$.

$$\therefore \text{Amplitude} = |a| = |-4| = 4$$

$$\text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{|1|} = 2\pi$$



```
Plot1 Plot2 Plot3
\Y1■4cos(X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



Window Settings:

$Xmin=-2\pi$

$Xmax=2\pi$

$Xscl=\frac{\pi}{6}$

$Ymin=-4$

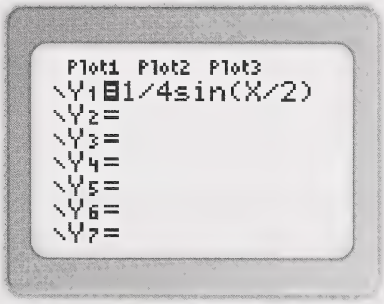
$Ymax=4$

$Yscl=0.5$

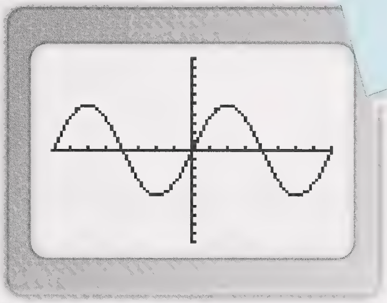
48. Compare $y=\frac{1}{4}\sin\frac{x}{2}$ with $y=a\sin bx$.

$$\therefore \text{Amplitude} = |a| = \left|\frac{1}{4}\right| = \frac{1}{4}$$

$$\text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{\left|\frac{1}{2}\right|} = 4\pi$$



```
Plot1 Plot2 Plot3
\Y1■1/4sin(X/2)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



Window Settings:

$Xmin=-4\pi$

$Xmax=4\pi$

$Xscl=\frac{\pi}{2}$

$Ymin=-0.5$

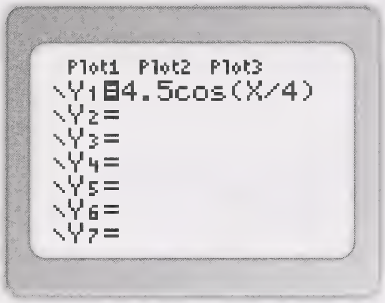
$Ymax=0.5$

$Yscl=0.05$

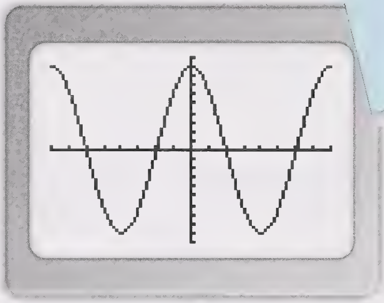
Section 1: Activity 3 (continued)

50. Compare $y = 4.5 \cos \frac{x}{4}$ with $y = a \cos bx$.

$$\therefore \text{Amplitude} = |a| = |4.5| = 4.5 \qquad \text{Period} = \frac{2\pi}{|b|} = \frac{2\pi}{\left|\frac{1}{4}\right|} = 8\pi$$



```
Plot1 Plot2 Plot3
\Y1■4.5cos(X/4)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



Window Settings:

$Xmin = -8\pi$
$Xmax = 8\pi$
$Xscl = \pi$
$Ymin = -5$
$Ymax = 5$
$Yscl = 0.5$

b. Textbook questions 54, 55, 57, 58, 61, and 62 of "Applications and Problem Solving," pp. 210 and 211

54.

$$\text{Amplitude} = |a| \qquad 10 = |a| \qquad a = \pm 10$$

$$\text{Period} = \frac{2\pi}{|b|} \qquad \frac{1}{440} = \frac{2\pi}{|b|} \qquad |b| = \frac{2\pi}{\frac{1}{440}} \qquad b = \pm 880\pi$$

Therefore, the possible equations are $y = 10 \sin 880\pi t$, $y = -10 \sin 880\pi t$, $y = 10 \sin(-880\pi t)$, and $y = -10 \sin(-880\pi t)$.

55. **a.** The period for both $y = \sin t$ and $y = \cos t$ is 2π. Since the frequency is the reciprocal of the period, the frequency of these functions is $\frac{1}{2\pi}$ Hz.

b. For $y = -\sin 2t$,

$$\begin{aligned}\text{Period} &= \frac{2\pi}{|b|}\\ &= \frac{2\pi}{|2|}\\ &= \pi\end{aligned}$$

For $y = 4\cos\frac{t}{2}$,

$$\begin{aligned}\text{Period} &= \frac{2\pi}{|b|}\\ &= \frac{2\pi}{\left|\frac{1}{2}\right|}\\ &= 4\pi\end{aligned}$$

The frequency of $y = -\sin 2t$ is $\frac{1}{\pi}$ Hz, and the frequency of $y = 4\cos\frac{t}{2}$ is $\frac{1}{4\pi}$ Hz.

57. **a.** Compare $V(t) = 170 \sin 120\pi t$ with $y = a \sin bx$.

$$\begin{aligned}\therefore\ \text{Amplitude} &= |a|\\ &= |170|\\ &= 170\end{aligned}$$

$$\begin{aligned}\text{Period} &= \frac{2\pi}{|b|}\\ &= \frac{2\pi}{|120\pi|}\\ &= \frac{1}{60}\end{aligned}$$

b.
$$\begin{aligned}\text{Average voltage} &= \frac{\text{amplitude}}{\sqrt{2}}\\ &= \frac{170}{\sqrt{2}}\\ &\doteq 120\end{aligned}$$

The average voltage is approximately 120 volts.

c. The frequency is the reciprocal of the period. Because the period is $\frac{1}{60}$, the frequency is 60 Hz.

Section 1: Activity 3 (continued)

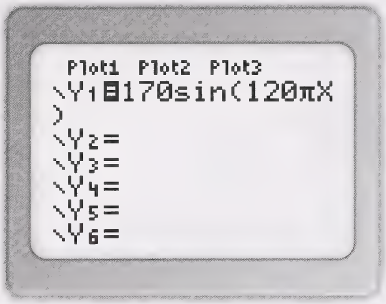
d. Calculate the voltage at the two times.

$$\begin{aligned} V(t) &= 170 \sin(120\pi t) \\ V(0.04) &= 170 \sin(120\pi \times 0.04) \\ &= 170 \sin 4.8\pi \\ &\doteq 99.92 \end{aligned}$$

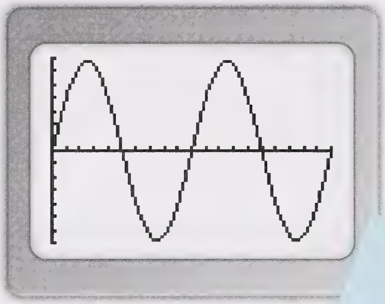
$$\begin{aligned} V(t) &= 170 \sin(120\pi t) \\ V(0.04) &= 170 \sin(120\pi \times 0.14) \\ &= 170 \sin 16.8\pi \\ &\doteq 99.92 \end{aligned}$$

The voltage did not change between these two times.

e.

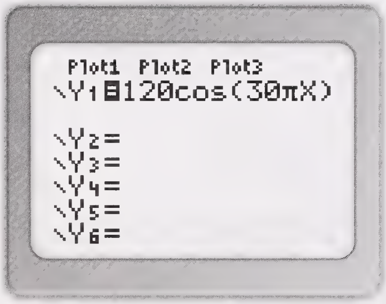


```
Plot1 Plot2 Plot3
\Y1■170sin(120πX
)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

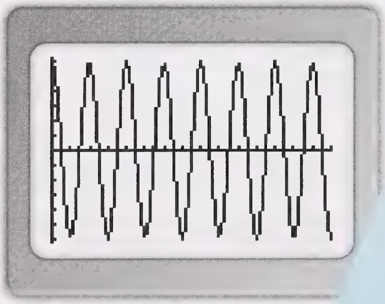


Window Settings:
$X\min = 0$
$X\max = \frac{2}{60}$
$Xscl = \frac{1}{600}$
$Y\min = -175$
$Y\max = 175$
$Yscl = 25$

58.



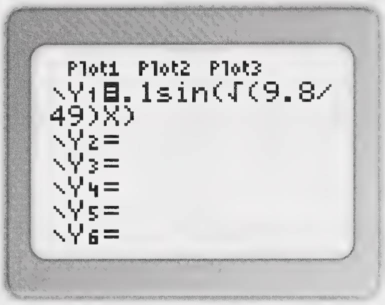
```
Plot1 Plot2 Plot3
\Y1■120cos(30πX)

\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



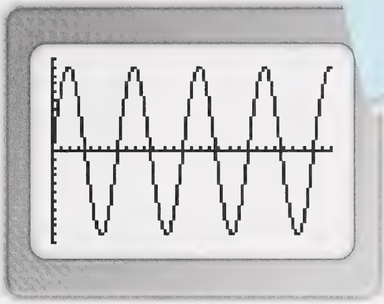
Window Settings:
$X\min = 0$
$X\max = 0.5$
$Xscl = \frac{0.5}{30}$
$Y\min = -125$
$Y\max = 125$
$Yscl = 20$

61. **a.** $\theta(t) = 0.1\sin\sqrt{\dfrac{9.8}{\ell}}t$

$$= 0.1\sin\sqrt{\dfrac{9.8}{49}}t$$



```
Plot1 Plot2 Plot3
\Y1■.1sin(√(9.8/
49)X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



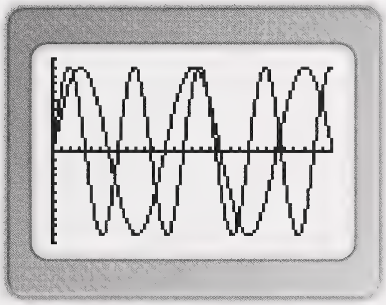
Window Settings:

Xmin = 0
Xmax = 60
Xscl = 2
Ymin = −0.11
Ymax = 0.11
Yscl = 0.01

The maximum value of θ is 0.1, the amplitude of the function.

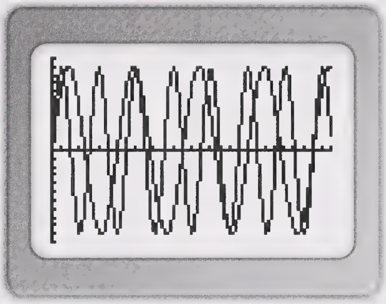
b. The maximum size of θ will remain at 0.1 if only ℓ changes; however, the period of the graph would increase as ℓ increases.

When the length was increased to 144 cm, the period increased.



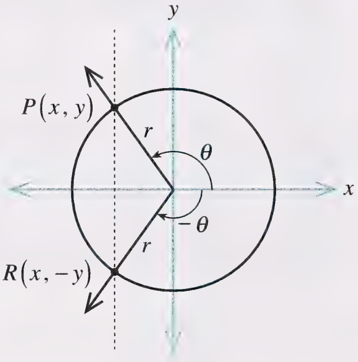
Section 1: Activity 3 (continued)

When the length was decreased to 16 cm, the period decreased.



Using $\theta(t) = 0.1 \sin\left(\sqrt{\frac{9.8}{\ell}}t\right)$ to describe the angle from the vertical for a pendulum leads to some rather interesting conclusions. The maximum angle is determined by the value of *a* in the sine equation; therefore, *a* is fixed at 0.1. The period of the pendulum's swing will vary with the length of the pendulum. The greater the length of the pendulum, the longer the period of its swing, and vice versa.

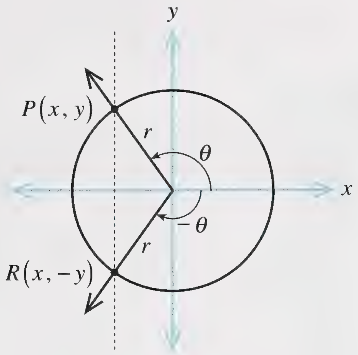
62. a. Sine is an odd function. This is most easily seen using the unit circle.



Consider *P* and *R*, points on the terminal arms of an angle and its negative. Because $\sin \theta = \frac{y}{r}$ and $\sin(-\theta) = \frac{-y}{r}$, $\sin(-\theta) = -\sin \theta$.

Therefore, $y = \sin \theta$ is an odd function.

b.



Consider P and R, points of the terminal arms of an angle and its negative. Because $\cos\theta = \frac{x}{r}$ and $\cos(-\theta) = \frac{x}{r}$, $\cos(-\theta) = \cos\theta$.

Therefore, $y = \cos\theta$ is an even function.

8. a. Textbook questions a. to e. of "Explore: Analyze a Graph," p. 212

a. The graphs all have a similar shape, rising and falling in a sinusoidal manner. The graphs either differ in period, amplitude, or placement on the plane.

b. The graph of $y = g(x)$ is the graph of $y = f(x)$ translated 2 units up.

c. The graph of $y = h(x)$ is the graph of $y = f(x)$ compressed vertically by a factor of $\frac{1}{2}$.

d. The graph of $y = j(x)$ is the graph of $y = f(x)$ translated $\frac{\pi}{4}$ to the right.

e. The graph of $y = k(x)$ is the graph of $y = f(x)$ expanded horizontally by a factor of $\frac{3}{2}$.

b. Textbook questions 1 to 6 of "Inquire," p. 212

1. $g(x) = \sin x + 2$ ⟵ translated 2 units up

$h(x) = \frac{1}{2}\sin x$ ⟵ compressed vertically by a factor of $\frac{1}{2}$

$j(x) = \sin\left(x - \frac{\pi}{4}\right)$ ⟵ translated $\frac{\pi}{4}$ to the right

$k(x) = \sin\frac{2}{3}x$ ⟵ expanded horizontally by a factor of $\frac{3}{2}$

Section 1: Activity 3 (continued)

2.

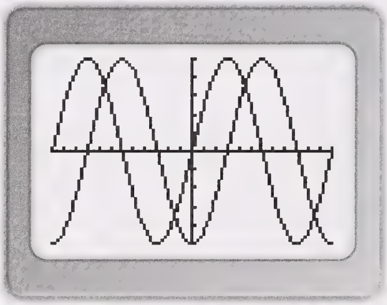
Graph	y-Intercept
$f(x) = \sin x$	0
$g(x) = \sin x + 2$	2
$h(x) = \frac{1}{2}\sin x$	0
$j(x) = \sin\left(x - \frac{\pi}{4}\right)$	$-\frac{\sqrt{2}}{2}$
$k(x) = \sin\frac{2}{3}x$	0

Remember: To find the y-intercept, let $x = 0$.

Each equation is of the form $y = a \sin bx + k$. The y-intercept of each graph is the value of k in its equation.

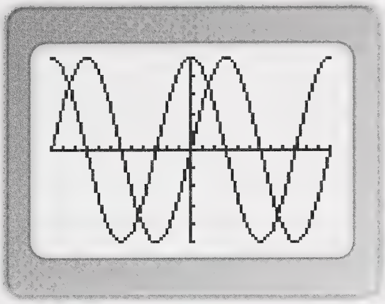
3. The graph $y = \sin x + k$ has no x-intercepts when the graph is completely above the x-axis. This will happen when $k > 1$. The graph of $y = \sin x + k$ also has no x-intercepts when the graph is completely below the x-axis. This will happen when $k < -1$.

4. The graph of Y_2 is the graph of Y_1 translated $\frac{\pi}{2}$ to the right.



5. The graph of $y = \sin\left(x + \frac{\pi}{2}\right)$ is the graph of $y = \sin x$ translated $\frac{\pi}{2}$ to the left.

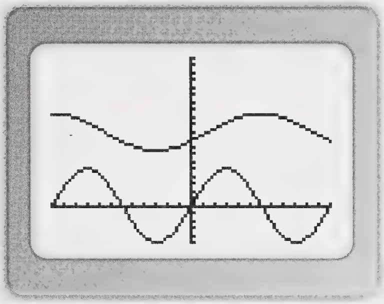
Check



The graph of $\cos x$ is the same graph as $\sin\left(x + \frac{\pi}{2}\right)$.

6. The graph of $y = 0.5 \sin \frac{2}{3}\left(x - \frac{\pi}{4}\right) + 2$ would have half the amplitude, have a period of 3π, and be translated $\frac{\pi}{4}$ to the right and 2 units up.

Check



9. Textbook questions 1, 3, 7, 8, 10, 13, and 15 of "Practice," p. 218

1. $C = 0$ and $D = 3$

   The horizontal phase shift is 0, and the vertical displacement is 3 units up.

3. $C = -45°$ and $D = 0$

   The horizontal phase shift is 45° to the right, and the vertical displacement is 0.

Section 1: Activity 3 (continued)

7. $C=\frac{3\pi}{8}$ and $D=-0.5$

The horizontal phase shift is $\frac{3\pi}{8}$ to the left, and the vertical displacement is 0.5 units down.

8. $C=-15°$ and $D=-4.5$

The horizontal phase shift is 15° to the right, and the vertical displacement is 4.5 units down.

10. $C=0$ and $D=-3$

The horizontal phase shift is 0, and the vertical displacement is 3 units down.

13. $C=-30°$ and $D=-2$

The horizontal phase shift is 30° to the right, and the vertical displacement is 2 units down.

15. $C=110°$ and $D=25$

The horizontal phase shift is 110° to the left, and the vertical displacement is 25 units up.

10. a. Textbook questions 17, 19, 23, 24, 26, 27, 29, 31, and 32 of "Practice," pp. 218 and 219

17. Compare $y=2\sin x-3$ to $y=A\sin B(x+C)+D$

$\therefore A=2, B=1, C=0,$ and $D=-3$

$$\begin{aligned}\text{Amplitude}&=|A|\\&=|2|\\&=2\end{aligned}$$

$$\begin{aligned}\text{Period}&=\frac{2\pi}{|B|}\\&=\frac{2\pi}{|1|}\\&=2\pi\end{aligned}$$

The horizontal phase shift is 0, and the vertical displacement is 3 units down.

19. Compare $y = \cos 3(x - 90°)$ to $y = A \cos B(x + C) + D$

$\therefore A = 1, B = 3, C = -90°$, and $D = 0$

$$\text{Amplitude} = |A| = |1| = 1 \qquad \text{Period} = \frac{360°}{|3|} = \frac{360°}{|3|} = 120°$$

The horizontal phase shift is 90° to the right, and the vertical displacement is 0.

23. Compare $y = -5 \sin 2\left(x - \frac{\pi}{6}\right) + 1$ to $y = A \sin B(x + C) + D$.

$\therefore A = -5, B = 2, C = -\frac{\pi}{6}$, and $D = 1$

$$\text{Amplitude} = |A| = |-5| = 5 \qquad \text{Period} = \frac{2\pi}{|B|} = \frac{2\pi}{|2|} = \pi$$

The horizontal phase shift is $\frac{\pi}{6}$ to the right, and the vertical displacement is 1 unit up.

24. Compare $y = 0.8 \cos \frac{2}{3}\left(x - \frac{\pi}{3}\right) - 7$ to $y = A \cos B(x + C) + D$.

$\therefore A = 0.8, B = \frac{2}{3}, C = -\frac{\pi}{3}$, and $D = -7$

$$\text{Amplitude} = |A| = |0.8| = 0.8 \qquad \text{Period} = \frac{2\pi}{|B|} = \frac{2\pi}{\left|\frac{2}{3}\right|} = 3\pi$$

The horizontal phase shift is $\frac{\pi}{3}$ to the right, and the vertical displacement is 7 units down.

Section 1: Activity 3 (continued)

26. Amplitude $= |A|$

$$2 = |A|$$
$$A = \pm 2$$

$$\text{Period} = \frac{2\pi}{|B|}$$
$$\pi = \frac{2\pi}{|B|}$$
$$|B| = \frac{2\pi}{\pi}$$
$$B = \pm 2$$

Because there is no horizontal phase shift, $C = 0$; because the vertical displacement is 1 unit upward, $D = 1$.

Therefore, the possible equations are $y = 2\cos 2x + 1$, $y = -2\cos 2x + 1$, $y = 2\cos(-2x) + 1$, and $y = -2\cos(-2x) + 1$.

27. Amplitude $= |A|$

$$1 = |A|$$
$$A = \pm 1$$

$$\text{Period} = \frac{360^\circ}{|B|}$$
$$360^\circ = \frac{360^\circ}{|B|}$$
$$|B| = \frac{360^\circ}{360^\circ}$$
$$B = \pm 1$$

Because the horizontal phase shift is 60° to the right, $C = -60^\circ$; because the vertical displacement is 3 units upward, $D = 3$.

Therefore, the possible equations are $y = \sin(x - 60^\circ) + 3$, $y = -\sin(x - 60^\circ) + 3$, $y = \sin[-(x - 60^\circ)] + 3$, and $y = -\sin[-(x - 60^\circ)] + 3$.

29. Amplitude $= |A|$

$$0.5 = |A|$$
$$A = \pm 0.5$$

$$\text{Period} = \frac{2\pi}{|B|}$$
$$2\pi = \frac{2\pi}{|B|}$$
$$|B| = \frac{2\pi}{2\pi}$$
$$B = \pm 1$$

Because the horizontal phase shift is $\frac{3\pi}{4}$ to the left, $C = \frac{3\pi}{4}$; because the vertical displacement is 2 units downward, $D = -2$.

Therefore, the possible equations are $y = 0.5 \cos\left(x + \frac{3\pi}{4}\right) - 2$, $y = -0.5 \cos\left(x + \frac{3\pi}{4}\right) - 2$, $y = 0.5 \cos\left[-\left(x + \frac{3\pi}{4}\right)\right] - 2$, and $y = -0.5 \cos\left[-\left(x + \frac{3\pi}{4}\right)\right] - 2$.

31. Amplitude $= \dfrac{|\max - \min|}{2}$

$$A = \frac{|1-(-3)|}{2}$$
$$= \frac{|4|}{2}$$
$$= 2$$

$$\text{Period} = \frac{2\pi}{|B|}$$
$$\frac{2\pi}{3} = \frac{2\pi}{|B|}$$
$$B = 3$$

Assume $B > 0$.

One period of the graph starts at $-\frac{\pi}{3}$. For $y = \sin x$, where $0 \le x \le 2\pi$, the period starts at 0. So, the curve has a horizontal phase shift of $\frac{\pi}{3}$ to the left; therefore, $C = \frac{\pi}{3}$.

The mid-line of the curve is the line $y = -1$. So, the vertical displacement of the curve is 1 unit down; therefore, $D = -1$.

The equation of the graph is $y = 2 \sin 3\left(x + \frac{\pi}{3}\right) - 1$.

Note: Answers may vary depending on the points selected on the graph.

Section 1: Activity 3 (continued)

32. Amplitude $= \dfrac{|\text{max} - \text{min}|}{2}$

$$\begin{aligned} A &= \frac{|7-(-3)|}{2} \\ &= \frac{|10|}{2} \\ &= 5 \end{aligned}$$

$$\begin{aligned} \text{Period} &= \frac{2\pi}{|B|} \\ 4\pi &= \frac{2\pi}{|B|} \\ |B| &= \frac{2\pi}{4\pi} \\ B &= \frac{1}{2} \end{aligned}$$

Assume $B > 0$.

One period of the graph starts at π. For $y = \sin x$, where $0 \le x \le 2\pi$, the period starts at 0. So, the curve has a horizontal phase shift of π to the right; therefore, $C = -\pi$.

The mid-line of the curve is the line $y = 2$. So, the vertical displacement of the curve is 2 units up; therefore, $D = 2$.

The equation of the graph is $y = 5 \sin \frac{1}{2}(x - \pi) + 2$.

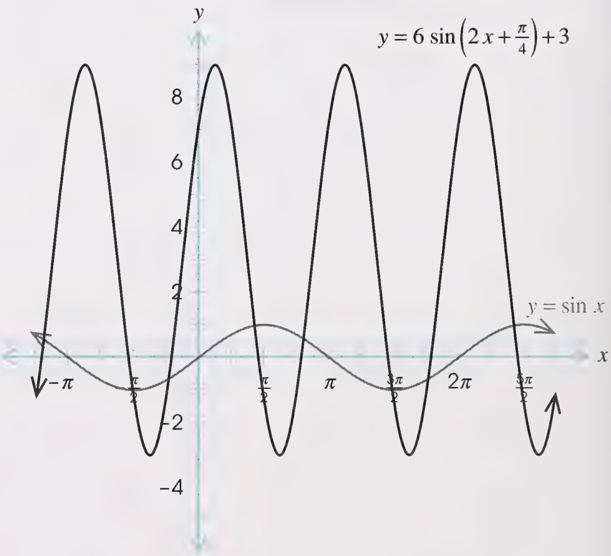
Note: Answers may vary depending on the points selected on the graph.

b. Textbook questions 34, 35, 37, and 38 of "Applications and Problem Solving," p. 219

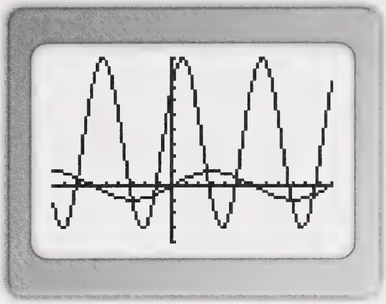
34. a. $y = 6 \sin\left(2x + \dfrac{\pi}{4}\right) + 3$

$$= 6 \sin 2\left(x + \frac{\pi}{8}\right) + 3$$

The amplitude is 6; the period is π; the phase shift is $\frac{\pi}{8}$ to the left; and the vertical displacement is 3 units up.



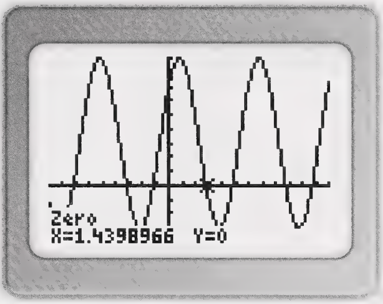
Check the graph using your graphing calculator.



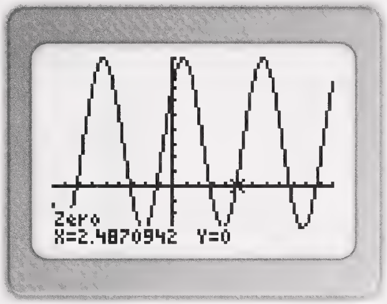
To find the y-intercept, let $x = 0$.

$$\begin{aligned} y &= 6\sin\left(2x + \frac{\pi}{4}\right) + 3 \\ &= 6\sin\left(\frac{\pi}{4}\right) + 3 \\ &\doteq 7.24 \end{aligned}$$

To find the x-intercepts, use the Zero feature on your graphing calculator.



```
Zero
X=1.4398966  Y=0
```

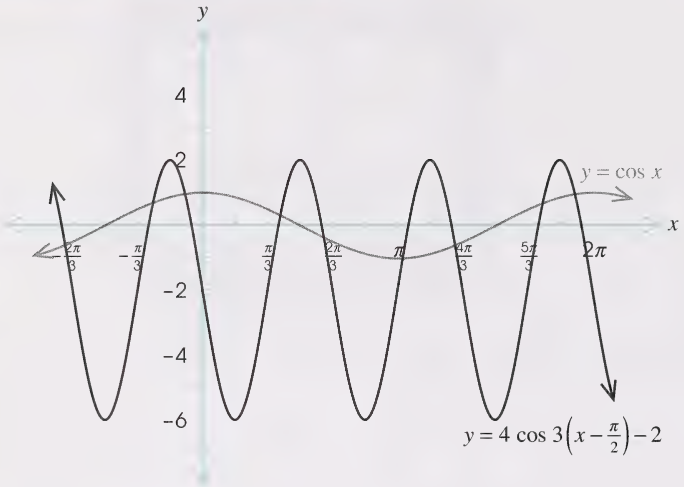


```
Zero
X=2.4870942  Y=0
```

The x-intercepts are approximately 1.44 and 2.49.

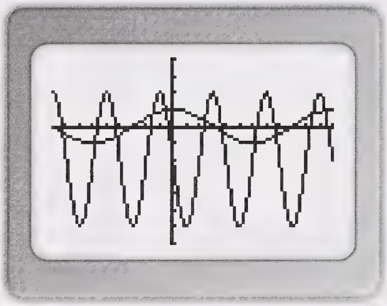
Section 1: Activity 3 (continued)

b.



The amplitude is 4; the period is $\frac{2\pi}{3}$; the phase shift is $\frac{\pi}{2}$ to the right; and the vertical displacement is 2 units down.

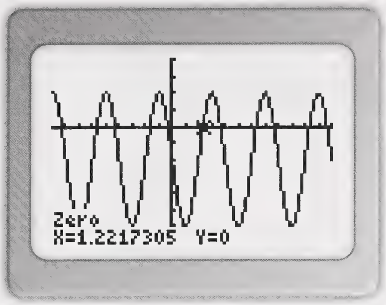
Check the graph using your graphing calculator.



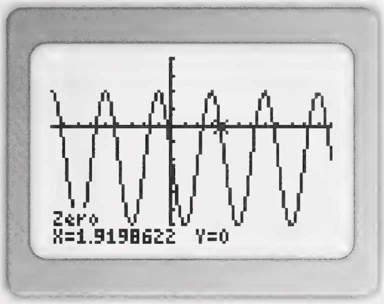
To find the y-intercept, let $x = 0$.

$$
\begin{aligned}
y &= 4\cos 3\left(x - \frac{\pi}{2}\right) - 2 \\
&= 4\cos 3\left(-\frac{\pi}{2}\right) - 2 \\
&= 4\cos\left(-\frac{3\pi}{2}\right) - 2 \\
&= -2
\end{aligned}
$$

Use the Zero feature to determine the *x*-intercepts.



```
Zero
X=1.2217305  Y=0
```

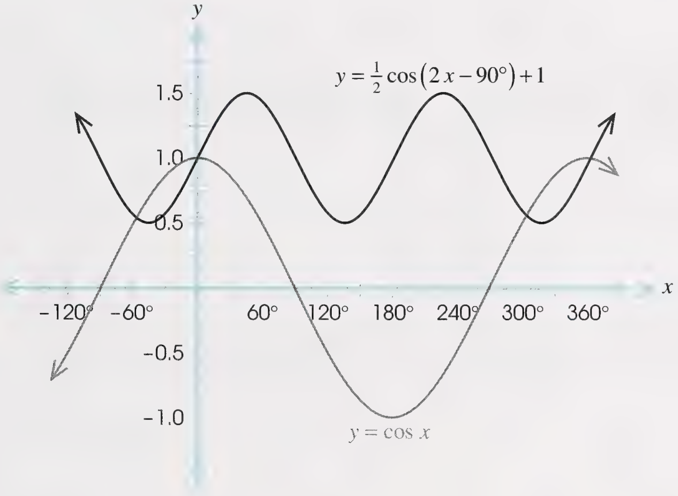


```
Zero
X=1.9198622  Y=0
```

The *x*-intercepts are approximately 1.22 and 1.92.

c. $y = \frac{1}{2}\cos\left(2x - 90°\right) + 1$

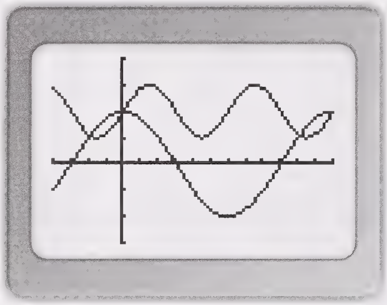
$= \frac{1}{2}\cos 2\left(x - 45°\right) + 1$



The amplitude is $\frac{1}{2}$ (or 0.5); the period is 180°; the phase shift is 45° to the right; and the vertical displacement is 1 unit up.

Section 1: Activity 3 (continued)

Check the graph using your graphing calculator.

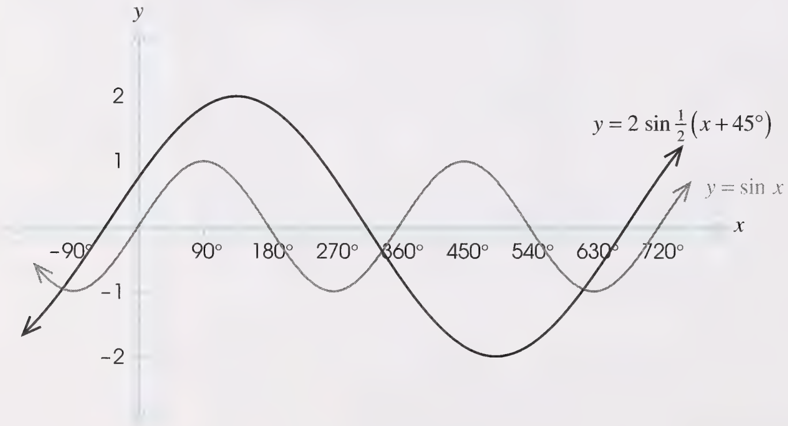


To find the y-intercept, let $x = 0$.

$$\begin{aligned} y &= \frac{1}{2}\cos(2x - 90^\circ) + 1 \\ &= \frac{1}{2}\cos(-90^\circ) + 1 \\ &= 1 \end{aligned}$$

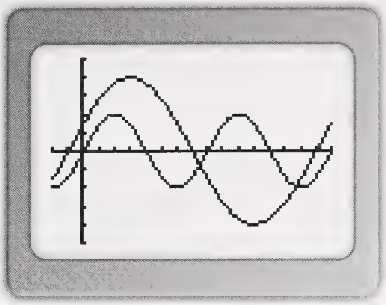
There are no x-intercepts.

d.



The amplitude is 2; the period is 720°; the phase shift is 45° to the left; and the vertical displacement is 0.

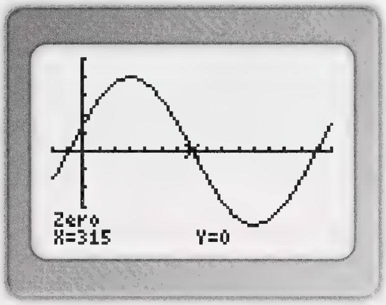
Check the graph using your graphing calculator.



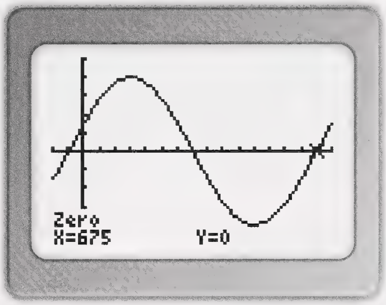
To find the *y*-intercept, let $x = 0$.

$$\begin{aligned} y &= 2\sin\frac{1}{2}(x+45^\circ) \\ &= 2\sin 22.5^\circ \\ &\doteq 0.77 \end{aligned}$$

Use the Zero feature to find the *x*-intercepts.



```
Zero
X=315          Y=0
```



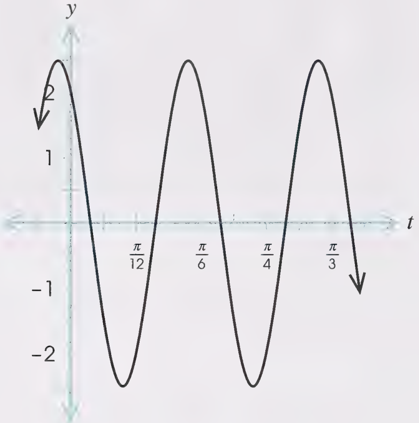
```
Zero
X=675          Y=0
```

The *x*-intercepts are 315° and 675°.

Section 1: Activity 3 (continued)

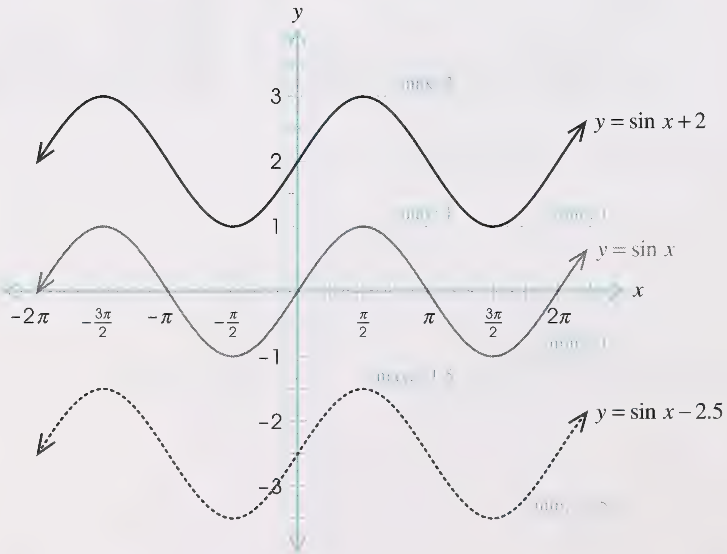
35. **a.** $y = 2.4 \cos\left(12t + \frac{\pi}{6}\right)$

$= 2.4 \cos 12\left(t + \frac{\pi}{72}\right)$



b. The maximum distance through which the object oscillates is 4.8 m, twice the amplitude.

37. The value of D in the sine function $y = A \sin B(x + C) + D$ does not affect the amplitude or the period of the function. The value of D, however, does affect the maximum and minimum values.



38. Since sine and cosine graphs have the same shape but differ in horizontal placement, you can find a value for C that makes the two graphs coincide. This phase shift is $-\frac{\pi}{2}$. Therefore, the graphs will be the same if $E = A$, $F = B$, $H = D$, and $G = C - \frac{\pi}{2}$. You could add $2k\pi$, where k is an integer, to G and still have the same graph since the period of the sine and cosine functions is 2π.

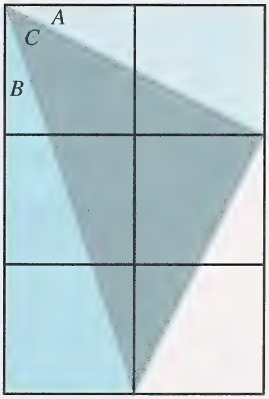
Since the amplitude is given by $|A|$, you would expect to find another set of values to make the graphs coincide when the first coefficients are negatives of each other. The phase shift for this is $\frac{\pi}{2}$. Therefore, the graphs will be the same if $E = -A$, $F = B$, $H = D$, and $G = C + \frac{\pi}{2}$. You could add $2k\pi$, where k is an integer, to G and still have the same graph.

Since the period is affected by $|B|$, you would expect that the graphs would coincide if the sign of B was changed. Therefore, the graphs will be the same if $E = A$, $F = -B$, $H = D$, and $G = C - \frac{\pi}{2} + 2k\pi$, where k is an integer; and the graphs will be the same if $E = -A$, $F = -B$, $H = D$, and $G = C + \frac{\pi}{2} + 2k\pi$, where k is an integer.

These conditions are summarized in the following table.

$E = A$	$F = B$	$H = D$	$G = C - \frac{\pi}{2} + 2k\pi$
$E = -A$	$F = B$	$H = D$	$G = C + \frac{\pi}{2} + 2k\pi$
$E = A$	$F = -B$	$H = D$	$G = C - \frac{\pi}{2} + 2k\pi$
$E = -A$	$F = -B$	$H = D$	$G = C + \frac{\pi}{2} + 2k\pi$

11. There are many ways to show that $\angle A + \angle B + \angle C = \frac{\pi}{2}$ or 90°. A sample answer is given.



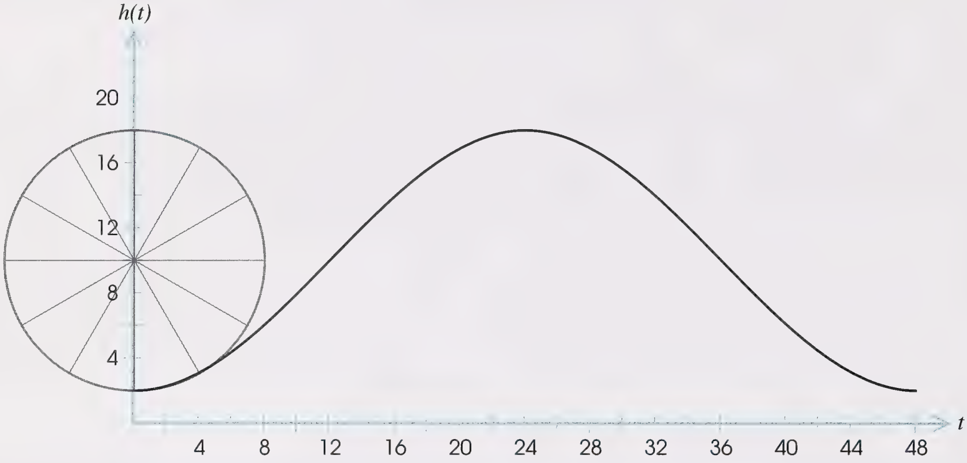
Draw a 2×3 rectangle.

From the diagram, you see that $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, and $\tan C = 1$.

Because the three angles fill one corner of the rectangle, $\angle A + \angle B + \angle C = 90°$ or $\frac{\pi}{2}$.

Section 1: Activity 4

1. **a. Textbook question "Explore: Draw a Graph," p. 222**



b. Textbook questions 1 to 6 of "Inquire," p. 222

1. Joy reaches a maximum height of 18 m and a minimum height of 2 m above ground. The amplitude of the graph is $\frac{18-2}{2} = 8$ m.

2. Joy's height above ground can be represented by a periodic function because her height repeats with each complete rotation of the Ferris wheel. The period is the time (in seconds) it takes for one rotation of the Ferris wheel, which is 48 s.

3. The amplitude is 8 m; so, $A = 8$.

   The period is 48 s.

$$\therefore \text{ Period} = \frac{2\pi}{|B|}$$

$$48 = \frac{2\pi}{|B|}$$

$$|B| = \frac{2\pi}{48}$$

$$B = \frac{\pi}{24} \leftarrow \text{[illegible]}$$

The graph crosses the mid-line at $t = 12$. Thus, the phase shift is 12 s to the right; so, $C = -12$.

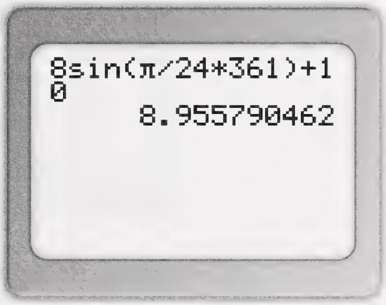
The whole wheel is 2 m above ground and the centre of the wheel is an additional 8 m above that. Thus, the vertical displacement is 10 m up; so, $D = 10$.

Therefore, an equation of the graph is $h(t) = 8 \sin \frac{\pi}{24}(t-12) + 10$.

4. Use the equation $y = 8 \sin \frac{\pi}{24}(t-12) + 10$ to predict Joy's height above the ground after 6 min 13 s or 373 s.

$$\begin{aligned} h(t) &= 8 \sin \tfrac{\pi}{24}(t-12) + 10 \\ h(373) &= 8 \sin \tfrac{\pi}{24}(373-12) + 10 \\ &= 8 \sin \tfrac{\pi}{24}(361) + 10 \\ &\doteq 9.0 \end{aligned}$$

Joy will be about 9.0 m above the ground.



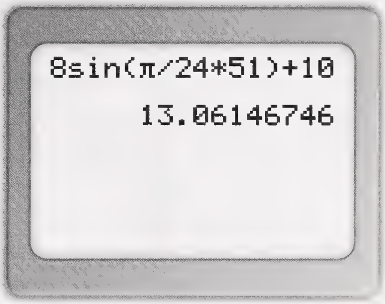
```
8sin(π/24*361)+1
0
         8.955790462
```

5. There are several ways to use the graphing calculator to find Joy's height above ground after 63 s.

Method 1: Using the Equation

$$\begin{aligned} h(t) &= 8 \sin \tfrac{\pi}{24}(t-12) + 10 \\ h(63) &= 8 \sin \tfrac{\pi}{24}(63-12) + 10 \\ &= 8 \sin \tfrac{\pi}{24}(51) + 10 \\ &\doteq 13.1 \end{aligned}$$

Joy is approximately 13.1 m above the ground.



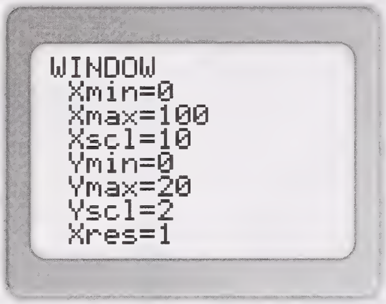
```
8sin(π/24*51)+10
         13.06146746
```

Section 1: Activity 4 (continued)

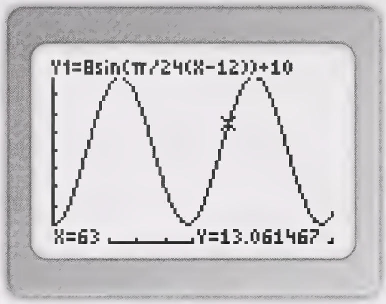
Method 2: Using the Graph

First, graph the function. Then press the following.

2nd [CALC] 1 (1:Value) 6 3 ENTER



```
WINDOW
 Xmin=0
 Xmax=100
 Xscl=10
 Ymin=0
 Ymax=20
 Yscl=2
 Xres=1
```

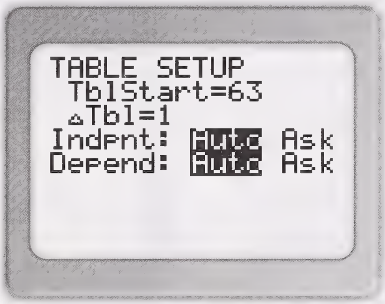


```
Y1=8sin(π/24(X-12))+10
X=63        Y=13.061467
```

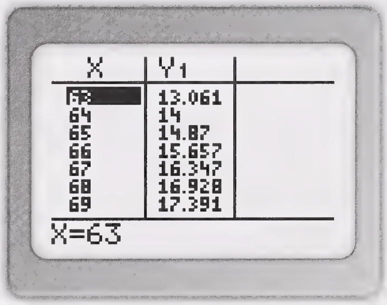
Joy is approximately 13.1 m above the ground.

Method 3: Using a Table of Values

First, graph the equation. Then press 2nd [TABLE] and find $x = 63$.



```
TABLE SETUP
 TblStart=63
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```



X	Y_1
63	13.061
64	14
65	14.87
66	15.657
67	16.347
68	16.928
69	17.391

X=63

Joy is approximately 13.1 m above the ground.

6. The diameter of the Ferris wheel is 100 m; so, $A = \frac{100}{2} = 50$.

The Ferris wheel has a period of 2 min or 120s.

$$\begin{aligned} \therefore\ \text{Period} &= \frac{2\pi}{|B|} \\ 120 &= \frac{2\pi}{|B|} \\ |B| &= \frac{2\pi}{120} \\ B &= \frac{\pi}{60} \quad \leftarrow \text{Assume } B > 0 \end{aligned}$$

The graph crosses the mid-line at $t = 30$. Thus, the phase shift is 30 s to the right; so, $C = -30$. The centre of the Ferris wheel is 55 m above ground; so, $D = 55$. Therefore, an equation that represents the Cosmoclock 21 Ferris wheel is $h(t) = 50 \sin \frac{\pi}{60}(x - 30) + 55$.

2. a. Textbook questions 1 to 3 and 6 to 15 of "Practice," pp. 225 and 226

1.

$$\begin{aligned} \text{Amplitude} &= \frac{|\max - \min|}{2} \\ A &= \frac{|4 - (-4)|}{2} \\ &= \frac{|8|}{2} \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{Period} &= \frac{2\pi}{|B|} \\ 2 &= \frac{2\pi}{|B|} \\ |B| &= \frac{2\pi}{2} \\ B &= \pi \end{aligned}$$

For a sine function, the phase shift is 0; so, $C = 0$. For a cosine function, the phase shift is 0.5 units to the right; so, $C = -0.5$.

The vertical displacement is 0; so, $D = 0$.

Therefore, the equations are $y = 4 \sin \pi t$ and $y = 4 \cos \pi(t - 0.5)$.

Section 1: Activity 4 (continued)

2. $$\text{Amplitude} = \frac{|\text{max} - \text{min}|}{2} \qquad \text{Period} = \frac{2\pi}{|B|}$$

$$A = \frac{|170 - (-170)|}{2} \qquad \frac{1}{60} = \frac{2\pi}{|B|}$$

$$= \frac{|340|}{2} \qquad |B| = 2\pi(60)$$

$$= 170 \qquad B = 120\pi$$

For a sine function, the phase shift is $\frac{1}{240}$ units to the left; so, $C = \frac{1}{240}$. For a cosine function, the phase shift is 0; so, $C = 0$.

The vertical displacement is 0; so, $D = 0$.

Therefore, the equations are $y = 170 \sin 120\pi\left(t + \frac{1}{240}\right)$ and $y = 170 \cos 120\pi t$.

3. $$\text{Amplitude} = \frac{|\text{max} - \text{min}|}{2} \qquad \text{Period} = \frac{2\pi}{|B|}$$

$$A = \frac{|6 - (-2)|}{2} \qquad 8 = \frac{2\pi}{|B|}$$

$$= \frac{|8|}{2} \qquad |B| = \frac{2\pi}{8}$$

$$= 4 \qquad B = \frac{\pi}{4}$$

For a sine function, the phase shift is 1 unit to the right; so, $C = -1$. For a cosine function, the phase shift is 3 units to the right; so, $C = -3$.

The vertical displacement is 2 units up; so, $D = 2$.

Therefore, the equations are $y = 4 \sin \frac{\pi}{4}(t - 1) + 2$ and $y = 4 \cos \frac{\pi}{4}(t - 3) + 2$.

6. The range is $-6.5 \le h(t) \le 6.5$.

7. $$\begin{aligned} A &= \frac{|\max - \min|}{2} \\ &= \frac{|6.5-(-6.5)|}{2} \\ &= \frac{|13|}{2} \\ &= 6.5 \end{aligned}$$

8. The period is 12 h.

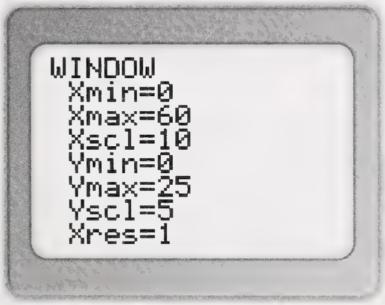
9. $$\begin{aligned} \text{Period} &= \frac{2\pi}{|B|} \\ 12 &= \frac{2\pi}{|B|} \\ |B| &= \frac{2\pi}{12} \\ B &= \frac{\pi}{6} \end{aligned}$$

10. Because a period starts at the origin, the phase shift is 0. Therefore, $C = 0$.

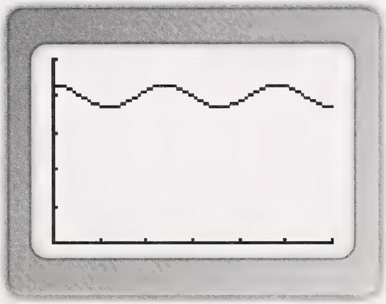
Because the mid-line is on the t-axis, the vertical displacement is 0. Therefore, $D = 0$.

11. The equation of the graph is $y = 6.5 \sin \frac{\pi}{6} t$.

12. Graph $y = 20 + 1.5 \cos \frac{\pi x}{12}$ on your graphing calculator.

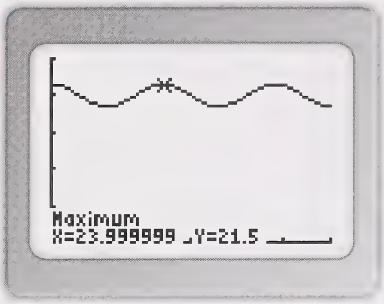


```
WINDOW
 Xmin=0
 Xmax=60
 Xscl=10
 Ymin=0
 Ymax=25
 Yscl=5
 Xres=1
```

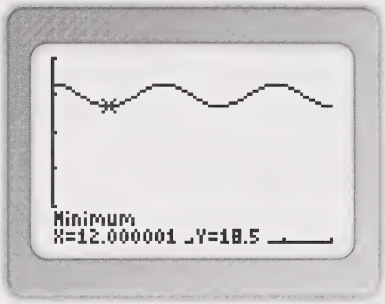


Section 1: Activity 4 (continued)

Use the maximum and minimum features.



Maximum
X=23.999999 Y=21.5



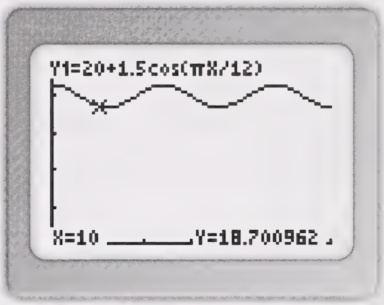
Minimum
X=12.000001 Y=18.5

The maximum value is 21.5°C, and the minimum value is 18.5°C.

13. $$t(x) = 20 + 1.5\cos\frac{\pi x}{12}$$

$$t(10) = 20 + 1.5\cos\frac{10\pi}{12}$$

$$\doteq 18.7$$



Y1=20+1.5cos(πX/12)

X=10 Y=18.700962

The temperature 10 min after the air conditioning turns on is approximately 18.7°C.

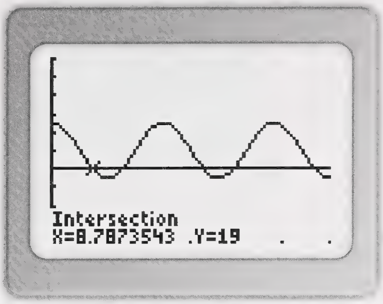
14. Compare $t(x) = 1.5\cos\frac{\pi}{12}x + 20$ with $y = A\sin B(x+C) + D$. Therefore, $B = \frac{\pi}{12}$.

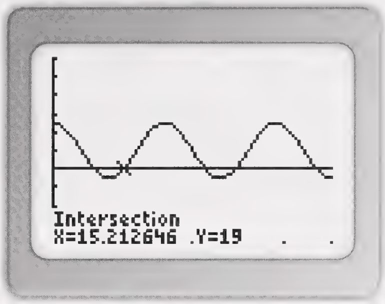
$$\therefore \text{ Period} = \frac{2\pi}{|B|}$$

$$= \frac{2\pi}{\frac{\pi}{12}}$$

$$= 24 \text{ min}$$

15. To determine the first two times the temperature is 19°C after the air conditioning turns on, find the points of intersection of the curve and the line $y = 19$.


Intersection
X=8.7873543 .Y=19


Intersection
X=15.212646 .Y=19

The temperature in the home is 19°C approximately 8.8 min and 15.2 min after the air conditioning turns on.

b. Textbook questions 17, 19, 21, and 24.a. of "Applications and Problem Solving," pp. 226 and 227

17. a. Compare $r(t) = 300 \sin \frac{\pi}{2} t + 1200$ with $y = A \sin B(x + C) + D$.

The vertical displacement, *D*, is 1200 units up. Therefore, the mid-line is at $r(t) = 1200$.

The amplitude, *A*, is 300.

The maximum and minimum values occur $|A|$ units from the mid-line.

$$\therefore \text{ maximum} = 1200 + 300 \qquad \text{minimum} = 1200 - 300$$
$$= 1500 \qquad\qquad = 900$$

b.
$$\begin{aligned}\text{Period} &= \frac{2\pi}{|B|}\\ &= \frac{2\pi}{\frac{\pi}{2}}\\ &= 4 \text{ years}\end{aligned}$$

Section 1: Activity 4 (continued)

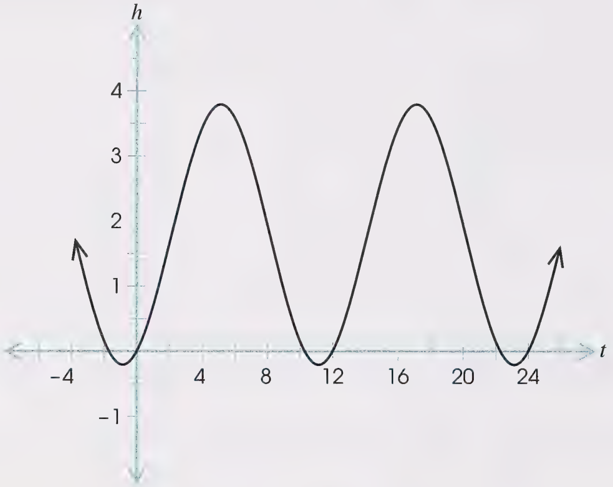
c. $r(t) = 1200 + 300 \sin \frac{\pi t}{2}$

$$\begin{aligned} r(40) &= 1200 + 300 \sin \frac{40\pi}{2} \\ &= 1200 + 300 \sin 20\pi \\ &= 1200 + 300(0) \\ &= 1200 \end{aligned}$$

In 2010, 40 years will have passed.

In 2010, 1200 rodents could be expected.

19. a. Answers may vary. A sample answer is given.



b. Amplitude is the same as the radius of the wheel; so, $A = 2$.

$$\begin{aligned} \text{Period} &= \frac{2\pi}{|B|} \\ \frac{60}{5} &= \frac{2\pi}{|B|} \\ |B| &= \frac{2\pi(5)}{60} \\ B &= \frac{\pi}{6} \text{ min} \end{aligned}$$

According to the graph in question 19.a., the phase shift is approximately 2.1 units to the right; so, $C \doteq -2.1$.

The mid-line occurs at $h = 1.8$. Therefore, the vertical displacement is 1.8 units up; so, $D = 1.8$.

An equation of the height of a point on the middle wheel is $h \doteq 2 \sin \frac{\pi}{6}(x - 2.1) + 1.8$.

21. **a.**

$$\begin{aligned} \text{Amplitude} &= \frac{|\max - \min|}{2} \\ A &= \frac{|26.3 - (-11.0)|}{2} \\ &= \frac{|37.3|}{2} \\ &= 18.65 \end{aligned}$$

$$\begin{aligned} \text{Period} &= \frac{2\pi}{|B|} \\ 12 &= \frac{2\pi}{|B|} \\ |B| &= \frac{2\pi}{12} \\ B &= \frac{\pi}{6} \end{aligned}$$

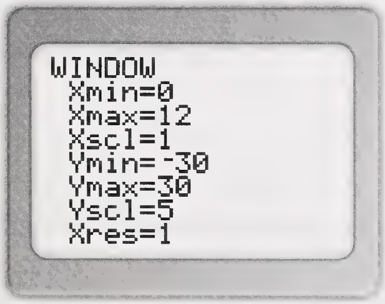
The maximum occurs in July. Therefore, for a cosine function, the phase shift is 7 units to the right; so, $C = -7$.

The vertical displacement, *D*, can be found by adding the amplitude to the minimum value or by subtracting the amplitude from the maximum value.

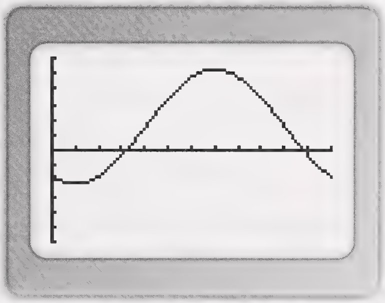
$$\begin{aligned} D &= -11.0 + 18.65 \\ &= 7.65 \end{aligned} \qquad \text{or} \qquad \begin{aligned} D &= 26.3 - 18.65 \\ &= 7.65 \end{aligned}$$

Therefore, an equation that approximates the data is $y = 18.65 \cos\left[\frac{\pi}{6}(x - 7)\right] + 7.65$.

b. Graph the function on your graphing calculator.



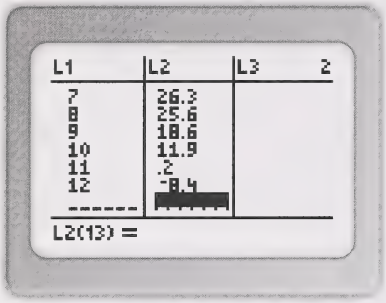
```
WINDOW
 Xmin=0
 Xmax=12
 Xscl=1
 Ymin=-30
 Ymax=30
 Yscl=5
 Xres=1
```



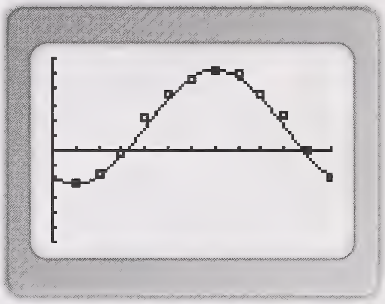
According to the graph, 5 of 12 months, which is about 41.7% of the year, average mid-afternoon temperatures below freezing.

Section 1: Activity 4 (continued)

c. Enter the data as a list into your graphing calculator, and plot the points together with the graph of the function from question 21.b.

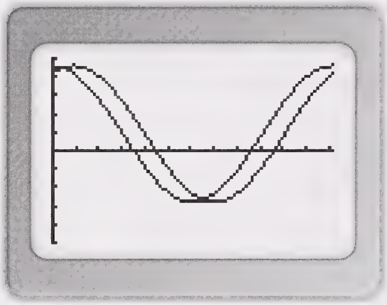


```
L1       L2       L3      2
 7        26.3
 8        25.6
 9        18.6
 10       11.9
 11       .2
 12       -8.4
 ------
L2(13) =
```



The model is a good approximation of the data.

24. a.



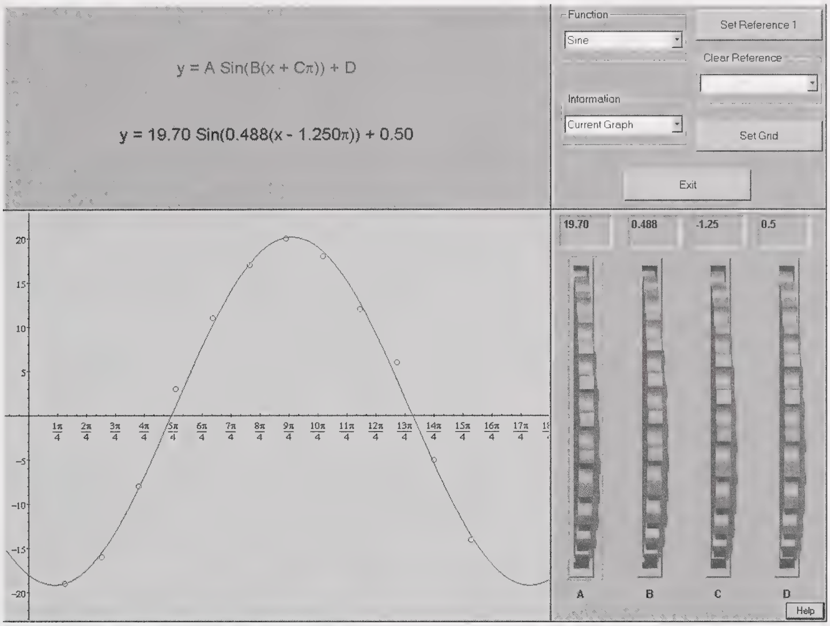
The graphs have the same amplitude, maximum, minimum, and period. However, they are different in horizontal location with the sine graph being translated to the left by about 30 days.

3. a. **Step 1:** Open the file titled *WinnipegTemp.txt*.

Step 2: Select the data points; then copy them.

Step 3: Open *The Trigonometric Functions Explorer*, and right-click the graph window. Then choose “Load ordered pairs from Clipboard.”

Step 4: Fit a sine curve to the data points. Your display should look similar to the following.



y = A Sin(B(x + Cπ)) + D

y = 19.70 Sin(0.488(x - 1.250π)) + 0.50

Function: Sine

Information: Current Graph

A: 19.70
B: 0.488
C: -1.25
D: 0.5

Note: Other possible equations are $y = 19.7 \sin\left[-0.488\left(x + 0.8\pi\right)\right] + 0.50$, $y = -19.7 \sin 0.488\left(x + 0.8\pi\right) + 0.50$, and $y = -19.7 \sin\left[-0.488\left(x - 1.25\pi\right)\right] + 0.50$.

b. For $x = 3$,

$$\begin{aligned} y &= 19.7 \sin 0.488\left(x - 1.25\pi\right) + 0.5 \\ &= 19.7 \sin 0.488\left(3 - 1.25\pi\right) + 0.5 \\ &\doteq -8.1°\text{C} \end{aligned}$$

For $x = 5$,

$$\begin{aligned} y &= 19.7 \sin 0.488\left(x - 1.25\pi\right) + 0.5 \\ &= 19.7 \sin 0.488\left(5 - 1.25\pi\right) + 0.5 \\ &\doteq 10.4°\text{C} \end{aligned}$$

For $x = 10$,

$$\begin{aligned} y &= 19.7 \sin 0.488\left(x - 1.25\pi\right) + 0.5 \\ &= 19.7 \sin 0.488\left(10 - 1.25\pi\right) + 0.5 \\ &\doteq 4.0°\text{C} \end{aligned}$$

The predicted temperatures obtained from the equation are close to the actual temperatures given. This equation is a good model for the monthly average temperatures in Winnipeg.

Section 1: Activity 4 (continued)

4. Answers may vary. Sample answers are given.

$1+2+34-5+67-8+9=100$
$123-4-5-6-7+8-9=100$
$123+45-67+8-9=100$
$12-3-4+5-6+7+89=100$
$1+23-4+5+6+78-9=100$
$1+2+3-4+5+6+78+9=100$
$12+3-4+5+67+8+9=100$
$123+4-5+67-89=100$
$123-45-67+89=100$
$12+3+4+5-6-7+89=100$
$1+23-4+56+7+8+9=100$

Section 1: Activity 5

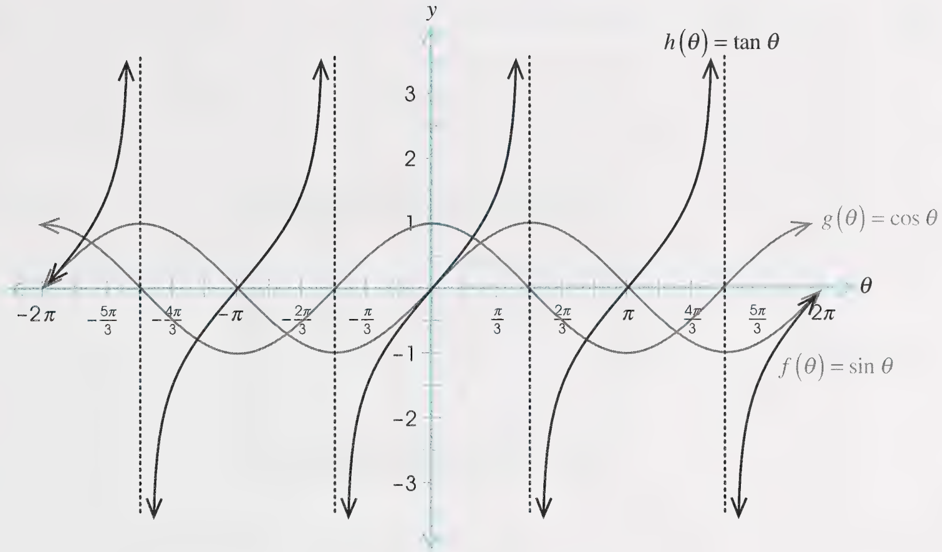
1. a. Textbook question "Explore: Use Skills With Functions," p. 228

The values of sine and cosine at $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π are values of special angles that you have memorized. They can be used along with CAST to rough in sketches of the sine and cosine functions.

	-2π	$-\frac{11\pi}{6}$	$-\frac{7\pi}{4}$	$-\frac{5\pi}{3}$	$-\frac{3\pi}{2}$	$-\frac{4\pi}{3}$	$-\frac{5\pi}{4}$	$-\frac{7\pi}{6}$	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{3\pi}{4}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	*	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	*	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	1
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	*	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	*	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0

* The calculation is undefined.



b. Textbook questions 1 to 8 of "Inquire," p. 228

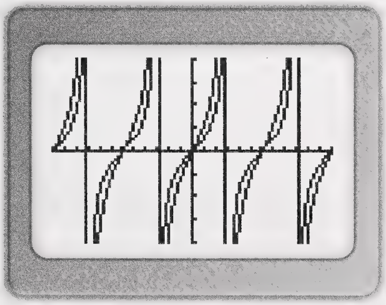
1. The domain of $f(\theta)$ and $g(\theta)$ is the set of real numbers. The domain of $h(\theta)$ is the set of real numbers, where $x \neq \frac{\pi}{2} + k\pi$ and k is any integer.

2. The period of $h(\theta)$ is π.

3. The range of $h(\theta)$ is the set of real numbers.

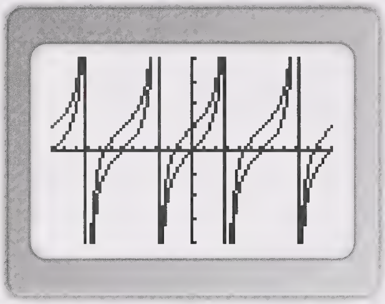
4. Because $h(\theta)$ has neither maximum nor minimum values, it has no amplitude.

5. The graph of $y = 2\tan\theta$ would be stretched vertically by a factor of 2, but otherwise it is similar to $h(\theta) = \tan\theta$. This is shown on the following display.

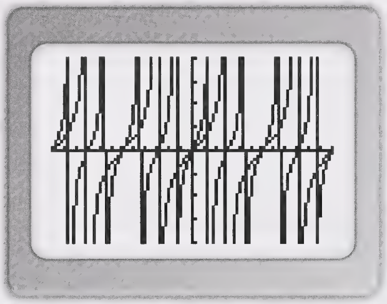


Section 1: Activity 5 (continued)

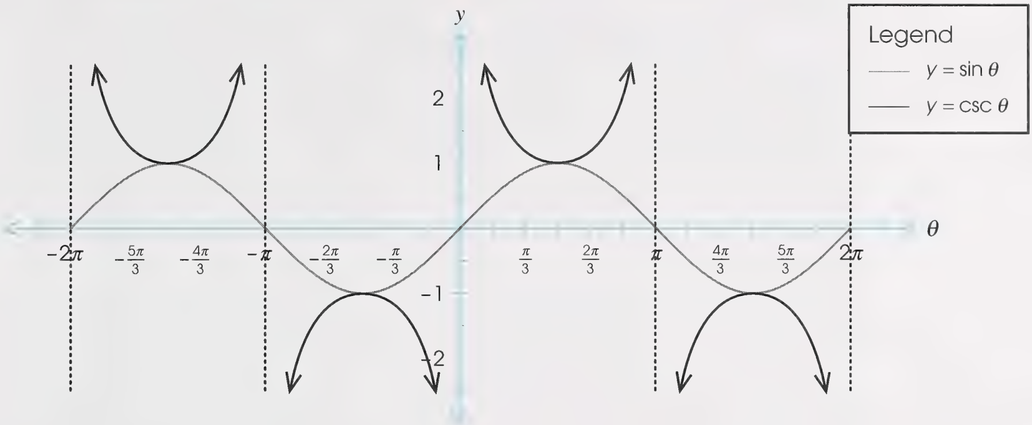
6. The graph of $y = \tan \theta + 1$ would be the graph of $h(\theta)$ translated 1 unit up. The graphs have the same shape and the locations of asymptotes are the same, but the x-intercepts are different.



7. The graph of $y = \tan 2\theta$ is the graph of $h(\theta)$ compressed horizontally by a factor or $\frac{1}{2}$. The period of $y = \tan 2\theta$ is half the period of $h(\theta)$.



8. The graph of $y = \csc \theta$ is the reciprocal of $y = f(\theta)$; so, for selected points on the graph of $y = f(\theta)$, find the reciprocal y-values and plot the resulting points as points on the graph of $y = \csc \theta$. Your graph should be similar to the following.



2. Textbook questions 1, 2, 4, 7, 9, and 10 of "Practice," p. 232

1. The graph has a period of $\frac{\pi}{2}$. Because the tangent function, $y = \tan x$, has a period of π, the value of k must be 2 in order to create a function with half the period. Therefore, the equation of the graph is $y = \tan 2x$.

2. The graph has a period of 2π. Because the tangent function, $y = \tan x$, has a period of π, the value of k must be $\frac{1}{2}$ in order to create a function with twice the period. Therefore, the equation of the graph is $y = \tan \frac{1}{2}x$.

4.

$$\begin{aligned} y &= \tan(3x - \pi) \\ &= \tan 3\left(x - \frac{\pi}{3}\right) \end{aligned}$$

The period is $\frac{\pi}{3}$, and the phase shift is $\frac{\pi}{3}$ units to the right.

7. $y = \tan\left(x - \frac{\pi}{4}\right)$

The period is π, and the phase shift is $\frac{\pi}{4}$ units to the right.

Section 1: Activity 5 (continued)

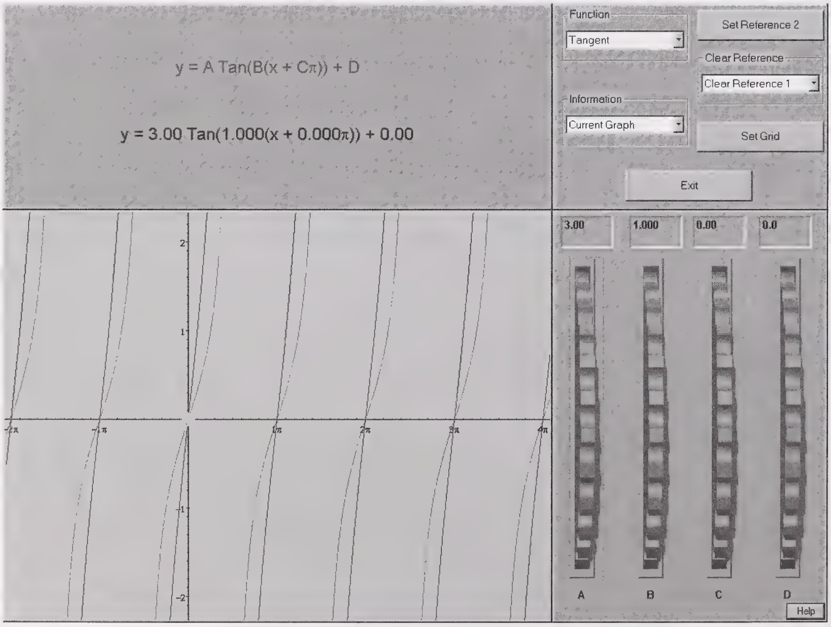
9. $$\begin{aligned} y &= \tan(2x - \pi) \\ &= \tan 2\left(x - \frac{\pi}{2}\right) \end{aligned}$$

The period is $\frac{\pi}{2}$, and the phase shift is $\frac{\pi}{2}$ units to the right.

10. $$\begin{aligned} y &= \tan\left(\frac{x}{2} + \frac{\pi}{2}\right) \\ &= \tan \frac{1}{2}(x + \pi) \end{aligned}$$

The period is $\frac{\pi}{\frac{1}{2}} = 2\pi$, and the phase shift is π units to the left.

3. a. When the value of *A* changes, the graph is compressed or stretched vertically. If $|A| > 1$, the graph is stretched vertically (becomes straighter). If $0 < |A| < 1$, the graph is compressed vertically (appears less steep at the *x*-intercepts). The period, domain, and range remain the same. For example, when the value of *A* changes from 1.0 to 3.0, the graph is stretched vertically by a factor of 3.



y = A Tan(B(x + Cπ)) + D

y = 3.00 Tan(1.000(x + 0.000π)) + 0.00

Function: Tangent

Clear Reference: Clear Reference 1

Information: Current Graph

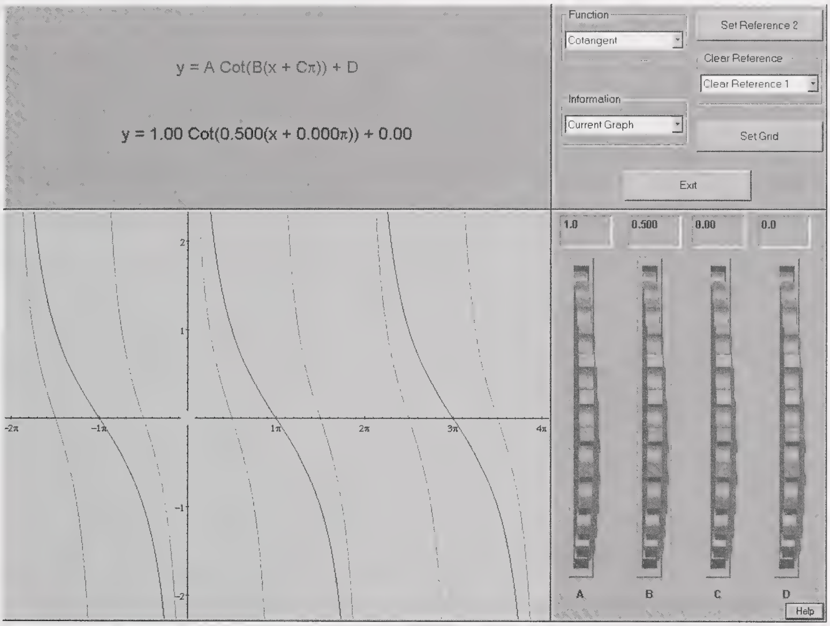
A: 3.00

B: 1.000

C: 0.00

D: 0.0

b. When the value of B changes, the period changes. If the value of B increases, the period decreases, and vice versa. For example, when the value of B changes from 1.0 to 0.5, the period changes from π to 2π.



y = A Cot(B(x + Cπ)) + D

y = 1.00 Cot(0.500(x + 0.000π)) + 0.00

Function: Cotangent

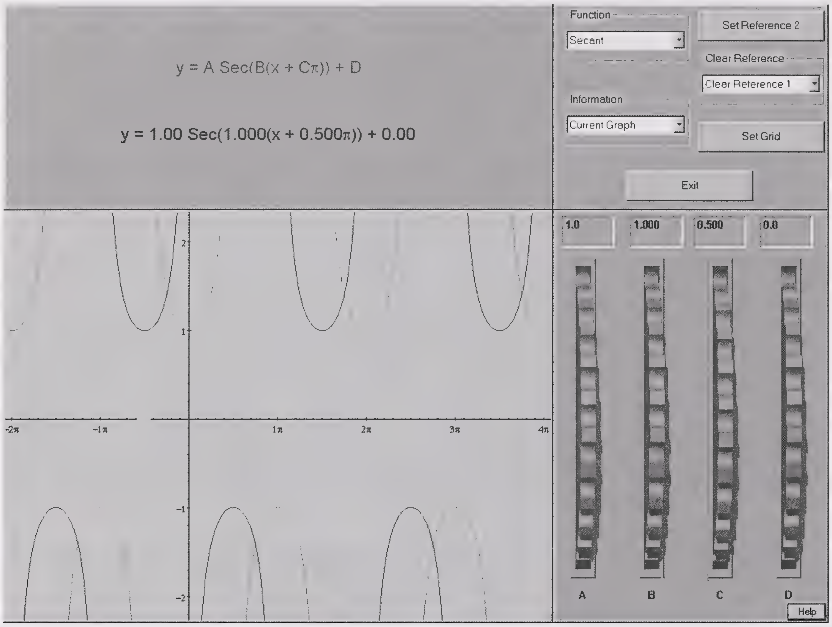
Information: Current Graph

Clear Reference: Clear Reference 1

A: 1.0
B: 0.500
C: 0.00
D: 0.0

Section 1: Activity 5 (continued)

c. When the value of C changes, the graph either moves to the left or to the right. If $C > 0$, the graph shifts to the left the corresponding amount. If $C < 0$, the graph shifts to the right the corresponding amount. For example, when the value of C changes from 0.0 to 0.5, the graph shifts 0.5π to the left.



y = A Sec(B(x + Cπ)) + D

y = 1.00 Sec(1.000(x + 0.500π)) + 0.00

Function: Secant

Information: Current Graph

Clear Reference: Clear Reference 1

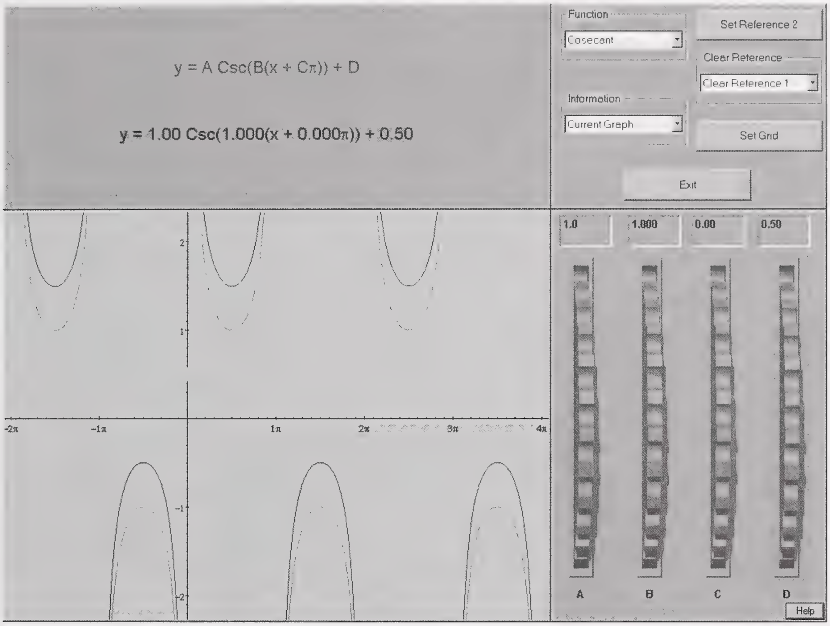
A: 1.0

B: 1.000

C: 0.500

D: 0.0

d. When the value of D changes, the graph either moves up or down. If $D > 0$, the graph moves up the corresponding amount. If $D < 0$, the graph moves down the corresponding amount. For example, when the value of D changes from 0.0 to 0.5, the graph moves up 0.5 units.



y = A Csc(B(x + Cπ)) + D

y = 1.00 Csc(1.000(x + 0.000π)) + 0.50

Function: Cosecant

Information: Current Graph

Clear Reference: Clear Reference 1

A: 1.0

B: 1.000

C: 0.00

D: 0.50

4. Textbook questions 16, 17, 19, 21, and 25 of "Applications and Problem Solving," p. 232

16. a. $\text{Period} = \dfrac{\pi}{|B|}$

$$2\pi = \frac{\pi}{|B|}$$

$$|B| = \frac{\pi}{2\pi}$$

$$B = \frac{1}{2}$$

The phase shift is $\frac{\pi}{4}$ to the left; so, $C = \frac{\pi}{4}$.

Therefore, an equation of the function is $y = \tan \frac{1}{2}\left(x + \frac{\pi}{4}\right)$.

Section 1: Activity 5 (continued)

b. There is an infinite number of equations. For example, $y = A \tan \frac{1}{2}\left(x + \frac{\pi}{4}\right) + D$, where $A \neq 0$ and D can be any number, represents a tangent function with a period of 2π and a phase shift of $\frac{\pi}{4}$ left.

17. a.
$$\begin{aligned} y &= \tan(2x - 90°) \\ &= \tan 2(x - 45°) \end{aligned}$$

$$\begin{aligned} \therefore \text{ Period} &= \frac{180°}{|B|} \\ &= \frac{180°}{2} \\ &= 90° \end{aligned}$$

The phase shift, C, is to the right.

Because the period of $y = \tan 2x$ is 90°, the first positive asymptote is at $x = 45°$. Because $y = \tan 2(x - 45°)$ is the graph of $y = \tan 2x$ shifted 45° to the right, the first positive asymptote is at $x = 90°$.

b. A general equation for all possible vertical asymptotes is $x = 90° n$, where n is any integer.

19. The graph of $y = \csc x$ is the graph of $y = \sec x$ translated $\frac{\pi}{2}$ to the right or left. Likewise the graph of $y = \sec x$ is the graph of $y = \csc x$ translated $\frac{\pi}{2}$ to the right or left.

21. a. Because $\cot x = \frac{1}{\tan x}$, $\tan x = \cot x$ when $\tan x = \pm 1$. This occurs when $x = \frac{\pi}{4} + k\frac{\pi}{2}$, where k is any integer.

Therefore, within the interval $-\pi \leq x \leq 2\pi$, $\tan x = \cot x$ when $x = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

b. Because $\cot x = \frac{1}{\tan x}$, $\cot x = -1$ when $\tan x = -1$. This occurs when $x = \frac{3\pi}{4} + k\pi$, where k is any integer.

Therefore, within the interval $-\pi \leq x \leq 2\pi$, $\cot x = -1$ when $x = -\frac{\pi}{4}, \frac{3\pi}{4}$, and $\frac{7\pi}{4}$.

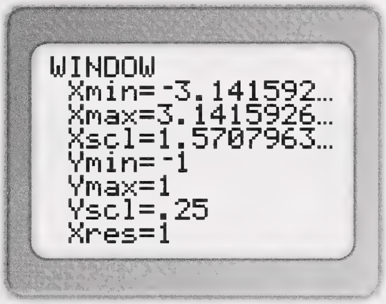
c. Because $\cot x = \frac{\cos x}{\sin x}$, $\cot x = 0$ when $\cos x = 0$. This occurs when $x = \frac{\pi}{2} + k\pi$, where k is any integer.

Therefore, within the interval $-\pi \le x \le 2\pi$, $\cot x = 0$ when $x = -\frac{\pi}{2}, \frac{\pi}{2}$, and $\frac{3\pi}{2}$.

d. $\cot x > \tan x$ wherever the graph of $y = \cot x$ is above the graph of $y = \tan x$. This occurs when $0 + k\frac{\pi}{2} < x < \frac{\pi}{4} + k\frac{\pi}{2}$, where k is any integer.

Therefore, within the interval $-\pi \le x \le 2\pi$, $\cot x > \tan x$ when $-\pi < x < -\frac{3\pi}{4}$, $-\frac{\pi}{2} < x < -\frac{\pi}{4}$, $0 < x < \frac{\pi}{4}$, $\frac{\pi}{2} < x < \frac{3\pi}{4}$, $\pi < x < \frac{5\pi}{4}$, and $\frac{3\pi}{2} < x < \frac{7\pi}{4}$.

25. To find values of x where $|\sec x| \ge |\tan x|$, graph $y = |\sec x| - |\tan x|$ on your graphing calculator. The portion of the graph that lies above the x-axis show the values of x where $|\sec x| \ge |\tan x|$.



```
WINDOW
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 Xmax=3.1415926…
 Xscl=1.5707963…
 Ymin=-1
 Ymax=1
 Yscl=.25
 Xres=1
```
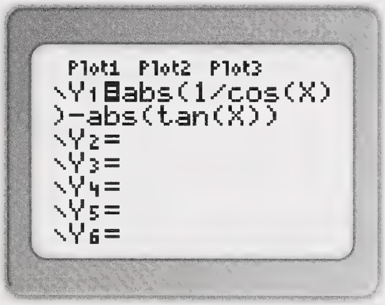
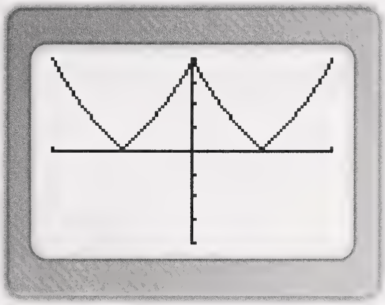



```
 Plot1 Plot2 Plot3
\Y1■abs(1/cos(X)
)-abs(tan(X))
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



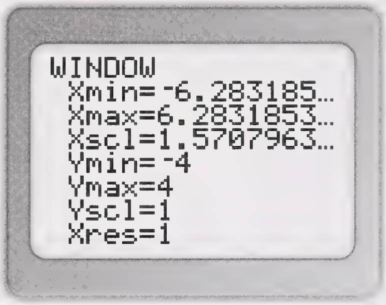
According to the graph, $|\sec x| \ge |\tan x|$ for all real values of x, where $x \ne \frac{\pi}{2} + k\pi$ and k is any integer.

Note: Secant and tangent are not defined at $x = \frac{\pi}{2} + k\pi$, where k is any integer.

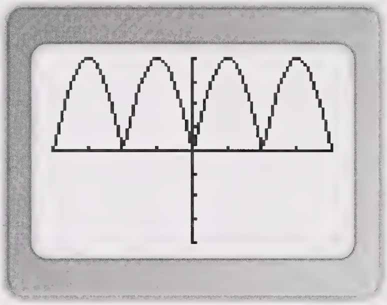
Section 1: Activity 5 (continued)

5. Textbook questions 4 and 6 of "Absolute Value and Trigonometric Functions," p. 233

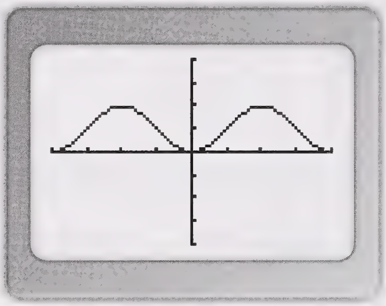
4. a. The graph of $y = |4 \sin x|$ has a period of π, a maximum of 4, and a minimum of 0. The y-intercept is 0, and the x-intercepts are $x = k\pi$, where k is any integer.



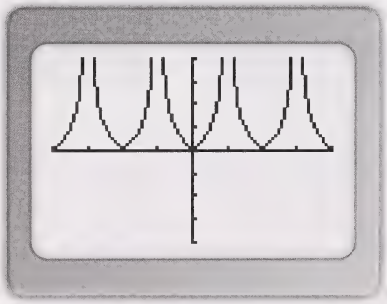
```
WINDOW
 Xmin=⁻6.283185…
 Xmax=6.2831853…
 Xscl=1.5707963…
 Ymin=⁻4
 Ymax=4
 Yscl=1
 Xres=1
```



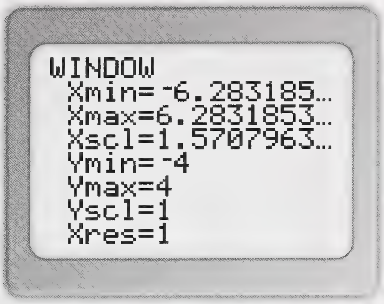
b. The graph of $y = |\cos x - 1|$ has a period of 2π, a maximum of 2, and a minimum of 0. The y-intercept is 0, and the x-intercepts are $x = 2\pi k$, where k is any integer.



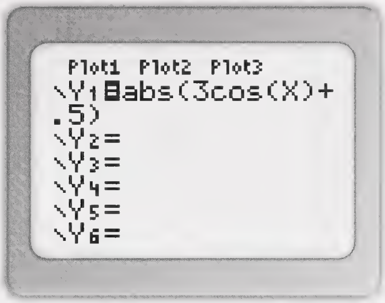
c. The graph of $y = |\tan x|$ has a period of π, no maximum, and a minimum of 0. The y-intercept is 0, and the x-intercepts are $x = k\pi$, where k is any integer.



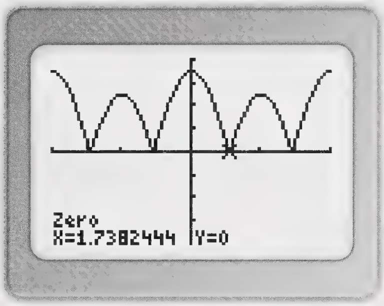
d. The graph of $y = \left|3 \cos x + \frac{1}{2}\right|$ has a period of 2π, a maximum of 3.5, and a minimum of 0. The y-intercept is 3.5. To find the x-intercepts, use the Zero feature on your graphing calculator.

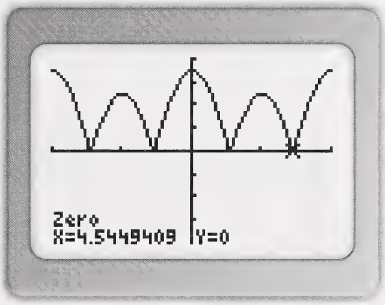


```
WINDOW
 Xmin=-6.283185…
 Xmax=6.2831853…
 Xscl=1.5707963…
 Ymin=-4
 Ymax=4
 Yscl=1
 Xres=1
```



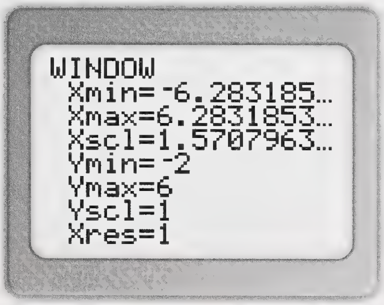
```
  Plot1  Plot2  Plot3
\Y1■abs(3cos(X)+
.5)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



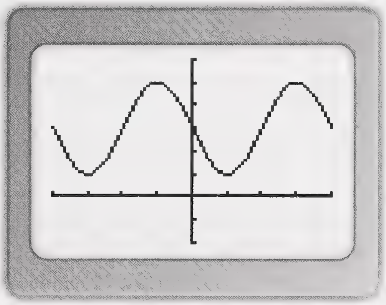


The x-intercepts are approximately $x = 1.74 + 2\pi k$ and $x = 4.54 + 2\pi k$, where k is any integer.

e. The graph of $y = |-2 \sin x + 3|$ has a period of 2π, a maximum of 5, and a minimum of 1. The y-intercept is 3, and there are no x-intercepts.

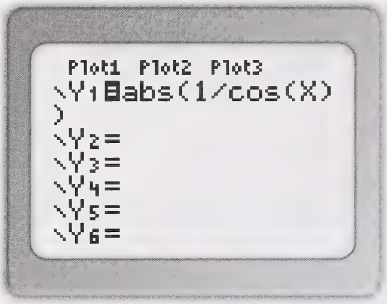


```
WINDOW
 Xmin=-6.283185…
 Xmax=6.2831853…
 Xscl=1.5707963…
 Ymin=-2
 Ymax=6
 Yscl=1
 Xres=1
```

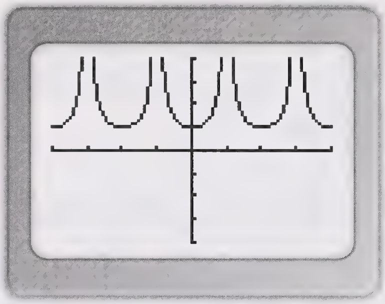


Section 1: Activity 5 (continued)

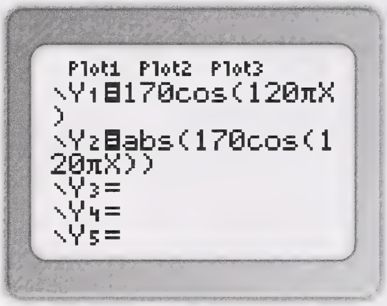
f. The graph of $y=|\sec x|$ has a period of π, no maximum, and a minimum of 1. The y-intercept is 1, and there are no x-intercepts.



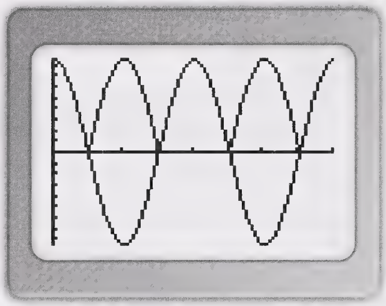
```
Plot1 Plot2 Plot3
\Y1▪abs(1/cos(X)
)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



6. a. Use your graphing calculator to sketch the graphs.



```
Plot1 Plot2 Plot3
\Y1▪170cos(120πX
)
\Y2▪abs(170cos(1
20πX))
\Y3=
\Y4=
\Y5=
```



b. Both graphs have a maximum of 170, a y-intercept at 170, and x-intercepts at $\frac{1}{240}+\frac{\pi}{120}k$, where k is any integer.

The graph of $V(t)=170\cos 120\pi t$ has a period of $\frac{1}{60}$ and a minimum of -170, whereas the graph of $V(t)=|170\cos 120\pi t|$ has a period of $\frac{1}{120}$ and a minimum of 0.

6. Textbook question "LOGIC POWER," p. 201

First, construct a table to help organize the information. Arrange the houses from left to right (first to last).

Door Colour	Nationality	Type of Vehicle	Type of Restaurant	Type of Pet

The Norwegian lives in the first house and lives next door to the house with the blue door.

The owner of the middle house (or third house) drives a car.

Because the person living in the house with the green door drives a truck and lives to the left of the house with the white door, the house with the green door must be the fourth house and the house with the white door must be the fifth house.

Your table should look like the following.

Door Colour	Nationality	Type of Vehicle	Type of Restaurant	Type of Pet
	[illegible]			
blue				
		[illegible]		
[illegible]		[illegible]		
[illegible]				

Section 1: Activity 5 (continued)

The British person lives in the house with the red door. So, the British person must live in the third house. This leaves the first house having a yellow door.

Your table should look like the following.

Door Colour	Nationality	Type of Vehicle	Type of Restaurant	Type of Pet
[illegible]	Norwegian			
blue				
[illegible]	[illegible]	car		
green		truck		
white				

The owner of the house with the yellow door prefers Chinese food.

The person living next door to the person who prefers Chinese food keeps horses. Thus, this person lives in the second house.

Your table should look like the following.

Door Colour	Nationality	Type of Vehicle	Type of Restaurant	Type of Pet
yellow	Norwegian		[illegible]	
blue				[illegible]
red	British	car		
green		truck		
white				

Because the Dane rides a motorcycle and the person who prefers Greek food rides a bicycle, the Norwegian must drive a van. Therefore, the person living in the house with the blue door must prefer Mexican food.

Because the person who prefers Greek food rides a bicycle, that person must live in the house with the white door. Therefore, the Dane, who drives a motorcycle, must live in the house with the blue door.

Your table should look like the following.

Door Colour	Nationality	Type of Vehicle	Type of Restaurant	Type of Pet
yellow	Norwegian		Chinese	
blue				horses
red	British	car		
green		truck		
white				

Because the German prefers East Indian food, the British person must prefer Italian food. Therefore, the German lives in the house with the green door and the Swede lives in the house with the white door.

The Swede keeps dogs, and the person who prefers Italian food has a parrot. Therefore, the Norwegian has a cat.

Your table should look like the following.

Door Colour	Nationality	Type of Vehicle	Type of Restaurant	Type of Pet
yellow	Norwegian	van	Chinese	
blue	Danish	motorcycle	Mexican	horses
red	British	car		
green		truck		
white		bicycle	Greek	

Section 1: Activity 5 (continued)

Therefore, the German owns the fish.

Door Colour	Nationality	Type of Vehicle	Type of Restaurant	Type of Pet
yellow	Norwegian	van	Chinese	cat
blue	Danish	motorcycle	Mexican	horses
red	British	car	Italian	parrot
green	German	truck	East Indian	fish
white	Swedish	bicycle	Greek	dogs

Section 1: Follow-up Activities

Extra Help

1. Textbook questions 32 to 43 of "Review," pp. 236 and 237

32. Compare $y = 4.2 \sin 3\theta$ with $y = A \sin B(\theta + C) + D$.

$\therefore\ A = 4.2$ and $B = 3$

$$\begin{aligned} \text{Amplitude} &= |A| \\ &= |4.2| \\ &= 4.2 \end{aligned} \qquad \begin{aligned} \text{Period} &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{|3|} \\ &= \frac{2\pi}{3} \end{aligned}$$

33. Compare $y = -2 \cos \frac{1}{3}\theta$ with $y = A \cos B(\theta + C) + D$.

$\therefore\ A = -2$ and $B = \dfrac{1}{3}$

$$\begin{aligned} \text{Amplitude} &= |A| \\ &= |-2| \\ &= 2 \end{aligned} \qquad \begin{aligned} \text{Period} &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\left|\frac{1}{3}\right|} \\ &= 6\pi \end{aligned}$$

34. Compare $y = 0.7 \cos \tfrac{3}{4}\theta$ with $y = A \cos B(\theta + C) + D$.

$\therefore A = 0.7$ and $B = \dfrac{3}{4}$

$$\begin{aligned} \text{Amplitude} &= |A| \\ &= |0.7| \\ &= 0.7 \end{aligned} \qquad \begin{aligned} \text{Period} &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\left|\frac{3}{4}\right|} \\ &= \frac{8\pi}{3} \end{aligned}$$

35. Compare $y = 5 \sin \tfrac{7}{12}\theta$ with $y = A \sin B(\theta + C) + D$.

$\therefore A = 5$ and $B = \dfrac{7}{12}$

$$\begin{aligned} \text{Amplitude} &= |A| \\ &= |5| \\ &= 5 \end{aligned} \qquad \begin{aligned} \text{Period} &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\left|\frac{7}{12}\right|} \\ &= \frac{24\pi}{7} \end{aligned}$$

36.

$$\begin{aligned} \text{Amplitude} &= \frac{|\max - \min|}{2} \\ A &= \frac{|4 - (-4)|}{2} \\ &= \frac{|8|}{2} \\ &= 4 \end{aligned} \qquad \begin{aligned} \text{Period} &= \frac{360°}{|B|} \\ 120° &= \frac{360°}{|B|} \\ |B| &= \frac{360°}{120°} \\ B &= 3 \end{aligned}$$

Because one period starts at the origin, there is no phase shift and there is no vertical displacement; so, $C = 0$ and $D = 0$.

Therefore, the equation of the graph is $y = 4 \sin 3x$.

Extra Help (continued)

37. $$\begin{aligned} \text{Amplitude} &= \frac{|\text{max} - \text{min}|}{2} \\ A &= \frac{|3-(-3)|}{2} \\ &= \frac{|6|}{2} \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{Period} &= \frac{2\pi}{|B|} \\ 4\pi &= \frac{2\pi}{|B|} \\ |B| &= \frac{2\pi}{4\pi} \\ B &= \frac{1}{2} \end{aligned}$$

Because one maximum occurs at $x=0$, there is no phase shift; so, $C=0$.

Because the mid-line lies along the x-axis, there is no vertical displacement; so, $D=0$.

Therefore, the equation is $y=3\cos\frac{x}{2}$.

38. Compare $y=2.5\sin x-1$ with $y=A\sin B(x+C)+D$.

$\therefore\ A=2.5,\ B=1,\ C=0$, and $D=-1$

$$\begin{aligned} \text{Amplitude} &= |A| \\ &= |2.5| \\ &= 2.5 \end{aligned}$$

$$\begin{aligned} \text{Period} &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{|1|} \\ &= 2\pi \end{aligned}$$

Because $C=0$, there is no phase shift.

Because $D=-1$, the vertical displacement is 1 unit down.

39. Compare $y = \cos 6(x - 45^\circ)$ with $y = A\cos B(x + C) + D$.

$\therefore A = 1, B = 6, C = -45^\circ$, and $D = 0$

$$\begin{aligned}\text{Amplitude} &= |A| \\ &= |1| \\ &= 1\end{aligned} \qquad \begin{aligned}\text{Period} &= \frac{360^\circ}{|B|} \\ &= \frac{360^\circ}{|6|} \\ &= 60^\circ\end{aligned}$$

Because $C = -45^\circ$, the phase shift is 45° to the right.

Because $D = 0$, there is no vertical displacement.

40. Compare $y = -4\cos 2\left(x + \frac{\pi}{3}\right) + 8$ with $y = A\cos B(x + C) + D$.

$\therefore A = -4, B = 2, C = \dfrac{\pi}{3}$, and $D = 8$

$$\begin{aligned}\text{Amplitude} &= |A| \\ &= |-4| \\ &= 4\end{aligned} \qquad \begin{aligned}\text{Period} &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{|2|} \\ &= \pi\end{aligned}$$

Because $C = \frac{\pi}{3}$, the phase shift is $\frac{\pi}{3}$ to the left.

Because $D = 8$, the vertical displacement is 8 units up.

41. Compare $y = 2\sin \frac{1}{3}(x + 90^\circ) + 4$ with $y = A\sin B(x + C) + D$.

$\therefore A = 2, B = \dfrac{1}{3}, C = 90^\circ$, and $D = 4$

$$\begin{aligned}\text{Amplitude} &= |A| \\ &= |2| \\ &= 2\end{aligned} \qquad \begin{aligned}\text{Period} &= \frac{360^\circ}{|B|} \\ &= \frac{360^\circ}{\left|\frac{1}{3}\right|} \\ &= 1080^\circ\end{aligned}$$

Because $C = 90^\circ$, the phase shift is 90° to the left.

Because $D = 4$, the vertical displacement is 4 units up.

Extra Help (continued)

42.

$$\begin{aligned}\text{Amplitude} &= \frac{|\text{max} - \text{min}|}{2}\\ A &= \frac{|2-(-6)|}{2}\\ &= \frac{|8|}{2}\\ &= 4\end{aligned}$$

$$\begin{aligned}\text{Period} &= \frac{360^\circ}{|B|}\\ 120^\circ &= \frac{360^\circ}{|B|}\\ |B| &= \frac{360^\circ}{120^\circ}\\ B &= 3\end{aligned}$$

For a sine function, the phase shift is 45° to the left; so, $C = 45^\circ$. For a cosine function, the phase shift is 15° to the left; so, $C = 15^\circ$.

Because the mid-line occurs at $y = -2$, the vertical displacement is 2 units down; so, $D = -2$.

Therefore, the equations of the graph are $y = 4 \sin 3(x + 45^\circ) - 2$ and $y = 4 \cos 3(x + 15^\circ) - 2$.

43.

$$\begin{aligned}\text{Amplitude} &= \frac{|\text{max} - \text{min}|}{2}\\ A &= \frac{|8-(-2)|}{2}\\ &= \frac{|10|}{2}\\ &= 5\end{aligned}$$

$$\begin{aligned}\text{Period} &= \frac{2\pi}{|B|}\\ 4\pi &= \frac{2\pi}{|B|}\\ |B| &= \frac{2\pi}{4\pi}\\ B &= \frac{1}{2}\end{aligned}$$

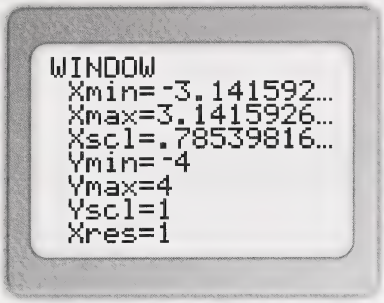
For a sine function, the phase shift is $\frac{2\pi}{3}$ to the left; so, $C = \frac{2\pi}{3}$. For a cosine function, the phase shift is $\frac{\pi}{3}$ to the right; so, $C = -\frac{\pi}{3}$.

Because the mid-line is at $y = 3$, the vertical displacement is 3 units up; so, $D = 3$.

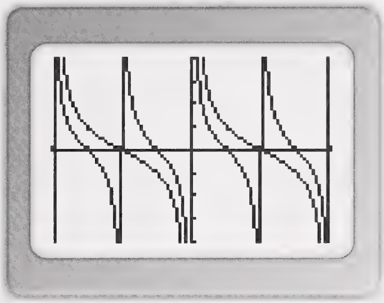
Therefore, the equations of the graph are $y = 5 \sin \frac{1}{2}\left(x + \frac{2\pi}{3}\right) + 3$ and $y = 5 \cos \frac{1}{2}\left(x - \frac{\pi}{3}\right) + 3$.

2. Textbook questions 50 and 51 of "Review," p. 237

50.



```
WINDOW
  Xmin=-3.141592...
  Xmax=3.1415926...
  Xscl=.78539816...
  Ymin=-4
  Ymax=4
  Yscl=1
  Xres=1
```

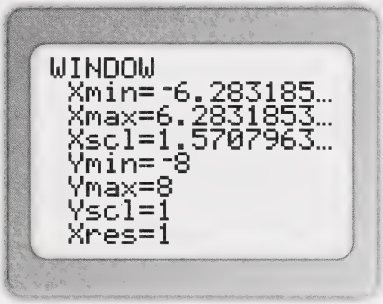


Using x-intercepts, the period of $y = \cot x$ is $\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$ and the period of $y = \cot 2x$ is $\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$.

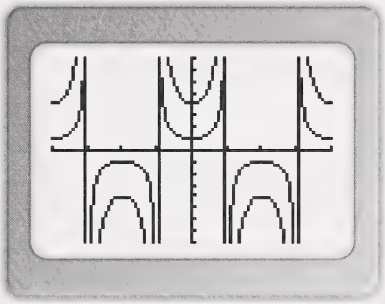
The domain of $y = \cot x$ is the set of real numbers except for $x = k\pi$, where k is any integer, and the domain of $y = \cot 2x$ is the set of real numbers except for $x = \frac{k\pi}{2}$, where k is any integer.

The range for both functions is the set of real numbers.

51.



```
WINDOW
  Xmin=-6.283185...
  Xmax=6.2831853...
  Xscl=1.5707963...
  Ymin=-8
  Ymax=8
  Yscl=1
  Xres=1
```



Using local maximums, the period of $y = \sec x$ is $\pi - (-\pi) = 2\pi$ and the period of $y = 4 \sec x$ is $\pi - (-\pi) = 2\pi$.

The domain of $y = \sec x$ and $y = 4 \sec x$ is the set of real numbers except for $x = \frac{\pi}{2} + k\pi$, where k is any integer.

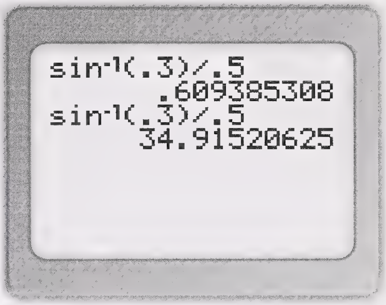
The range of $y = \sec x$ is the set of real numbers except those between -1 and 1; or the range of $y = \sec x$ is $y \geq 1$ and $y \leq -1$.

The range of $y = 4 \sec x$ is the set of real numbers except those between -4 and 4; or the range is $y \geq 4$ and $y \leq -4$.

Enrichment

1.

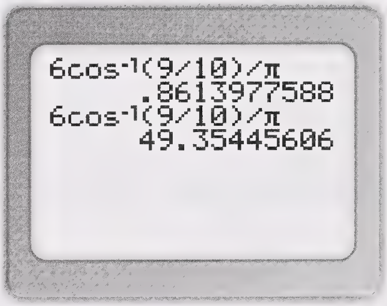
$$\begin{aligned} y &= 6\sin 0.5x - 1.8 \\ 0 &= 6\sin 0.5x - 1.8 \\ 6\sin 0.5x &= 1.8 \\ \sin 0.5x &= \frac{1.8}{6} \\ \sin 0.5x &= 0.3 \\ \sin^{-1}(\sin 0.5x) &= \sin^{-1}(0.3) \\ 0.5x &= \sin^{-1}(0.3) \\ x &= \frac{\sin^{-1}(0.3)}{0.5} \\ &\doteq 0.61 \text{ rad or } 34.9^\circ \end{aligned}$$



```
sin⁻¹(.3)/.5
          .609385308
sin⁻¹(.3)/.5
         34.91520625
```

2.

$$\begin{aligned} y &= -2\cos\frac{\pi}{6}x + \frac{9}{5} \\ 0 &= -2\cos\frac{\pi}{6}x + \frac{9}{5} \\ 2\cos\frac{\pi}{6}x &= \frac{9}{5} \\ \cos\frac{\pi}{6}x &= \frac{9}{10} \\ \cos^{-1}\left(\cos\frac{\pi}{6}x\right) &= \cos^{-1}\left(\frac{9}{10}\right) \\ \frac{\pi}{6}x &= \cos^{-1}\left(\frac{9}{10}\right) \\ x &= \frac{6\cos^{-1}\left(\frac{9}{10}\right)}{\pi} \\ &\doteq 0.86 \text{ rad or } 49.4^\circ \end{aligned}$$



```
6cos⁻¹(9/10)/π
          .8613977588
6cos⁻¹(9/10)/π
         49.35445606
```

Section 2: Activity 1

1. **a. Textbook questions a. and b. of "Explore: Use a Graph," p. 244**

   a. The x-intercepts of $f(x) = 2 \sin x$ in the domain $0 \le x < 360°$ are 0 and 180°.

   b. The values of x that make $f(x) = 1$ in the domain $0 \le x < 360°$ are 30° and 150°.

   b. Textbook questions 1 to 6 of "Inquire," p. 244

   1. The x-intercepts provide a solution to the equation $f(x) = 0$ or $2 \sin x = 0$.

   2. The equation $f(x) = 1$ or $2 \sin x = 1$ was solved by doing the interpolation.

   3. To find a solution for $2 = 2 \sin x$ or $f(x) = 2$, follow the graph until its value of y is 2. In the domain $0 \le x < 360°$, the solution is 90°.

   4. The minimum value of $f(x)$ in the domain $0 \le x < 360°$ occurs at 270°. This value provides the solution to $f(x) = -2$ or $2 \sin x = -2$.

   5. The graph tells you that there are no values in the solution to $3 = 2 \sin x$. The graph never rises above a y-value of 2.

   6. The graph of $f(x) = 2 \sin x$ has a value of -1 at the points where $\sin x$ has a value of -0.5. Thus, find the points (210° and 330°) where the graph takes the value -1.

2. **Textbook questions 2, 5, and 10 of "Practice," pp. 247 and 248**

   2. The graph of $y = \sin 2x$ is related to the graph of $y = \sin x$ by a horizontal compression factor of 2. In the interval $0 \le x < 2\pi$, $y = \sin x$ has one period; thus, $y = \sin 2x$ has two periods.

   The period of $y = \sin 2x$ is $\frac{2\pi}{2}$ or π.

   In the interval $0 \le x < 2\pi$, $\sin x = 0$ when $x = 0$ and π. Therefore, $\sin 2x = 0$ when $x = 0$ and $\frac{\pi}{2}$ in its first period.

   Recall that $y = \sin 2x$ has two periods in the interval $0 \le x < 2\pi$. In the second period, $\sin 2x = 0$ when $x = 0 + \pi$ and $\frac{\pi}{2} + \pi$, or π and $\frac{3\pi}{2}$.

   The solutions of $\sin 2x = 0$ in the interval $0 \le x < 2\pi$ are 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$.

Section 2: Activity 1 (continued)

5. The graph of $y = \sin 3x$ is related to the graph of $y = \sin x$ by a horizontal compression factor of 3. In the interval $0 \le x < 2\pi$, $y = \sin x$ has one period; thus, $y = \sin 3x$ has three periods.

The period of $y = \sin 3x$ is $\frac{2\pi}{3}$.

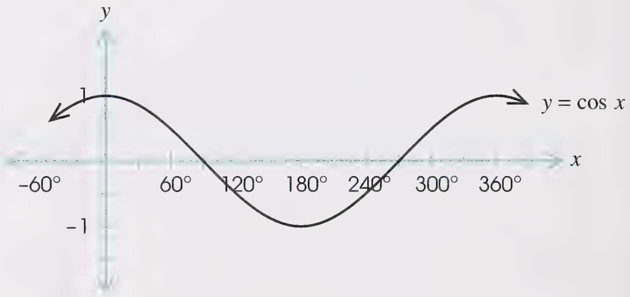
In the interval $0 \le x < 2\pi$, $\sin x = 1$ when $x = \frac{\pi}{2}$. Therefore, in its first period, $\sin 3x = 1$ when

$$\begin{aligned} x &= \frac{\frac{\pi}{2}}{3} \\ &= \frac{\pi}{6} \end{aligned}$$

Recall that $y = \sin 3x$ has three periods in the interval $0 \le x < 2\pi$. In the second period, $\sin 3x = 1$ when $x = \frac{\pi}{6} + \frac{2\pi}{3}$ or $\frac{5\pi}{6}$. In the third period, $\sin 3x = 1$ when $x = \frac{5\pi}{6} + \frac{2\pi}{3}$ or $\frac{3\pi}{2}$.

The solutions of $\sin 3x = 1$ in the interval $0 \le x < 2\pi$ are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, and $\frac{3\pi}{2}$.

10. The solutions to the equation $\cos x = 0$ are equivalent to finding the x-intercepts of the graph for the specified interval. From the graph, the x-intercepts of $y = \cos x$ occur at 90° and 270°.



The solutions of $\cos x = 0$ in the interval $0 \le x < 360°$ are 90° and 270°.

3. Textbook questions 3, 9, 12, and 15 of "Practice," pp. 247 and 248

3. The graph of $y = \sin 2x$ is related to the graph of $y = \sin x$ by a horizontal compression factor of 2. In the interval $0 \le x < 2\pi$, $y = \sin x$ has one period; thus, $y = \sin 2x$ has two periods.

The period of $y = \sin 2x$ is $\frac{2\pi}{2}$ or π.

In the interval $0 \le x < 2\pi$, $\sin x = 0$ when $x = 0$ and π. Therefore, $\sin 2x = 0$ when $x = 0$ and $\frac{\pi}{2}$ in its first period.

Because the period of $y = \sin 2x$ is π, all the solutions of $\sin 2x = 0$ are $x = \frac{k\pi}{2}$, where k is any integer.

9. The graph of $y = \sin 4x$ is related to the graph of $y = \sin x$ by a horizontal compression factor of 4. In the interval $0 \le x < 2\pi$, $y = \sin x$ has one period; thus, $y = \sin 4x$ has four periods.

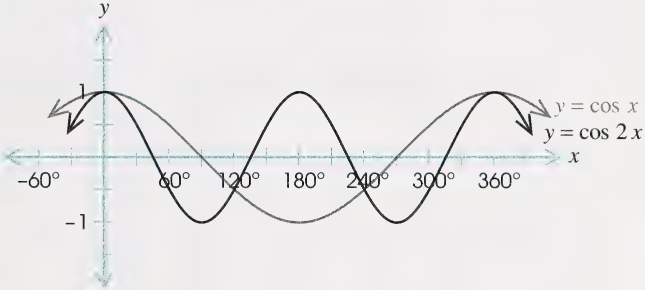
The period of $y = \sin 4x$ is $\frac{2\pi}{4}$ or $\frac{\pi}{2}$.

In the interval $0 \le x < 2\pi$, $\sin x = -1$ when $x = \frac{3\pi}{2}$. Therefore, in its first period, $\sin 4x = -1$ when

$$\begin{aligned} x &= \frac{\frac{3\pi}{2}}{4} \\ &= \frac{3\pi}{8} \end{aligned}$$

Because the period of $y = \sin 4x$ is $\frac{\pi}{2}$, all the solutions of $\sin 4x = -1$ are $\frac{3\pi}{8} + \frac{k\pi}{2}$, where k is any integer.

12. The graph of $y = \cos 2x$ is related to the graph of $y = \cos x$ by a horizontal compression factor of 2. In the interval $0 \le x < 360°$, $y = \cos x$ has one period; thus, $y = \cos 2x$ has two periods.



The period of $y = \cos 2x$ is $\frac{360°}{2}$ or 180°.

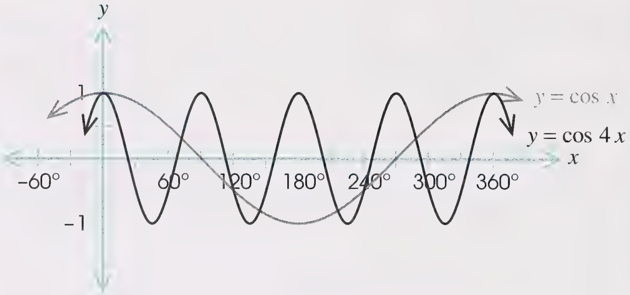
In the interval $0 \le x < 360°$, $\cos x = 0$ when $x = 90°$ and 270°. Therefore, $\cos 2x = 0$ when $x = 45°$ and 135° in its first period.

Because the period of $y = \cos 2x$ is 180°, all the solutions of $\cos 2x = 0$ are $x = 45° + 180° k$ and $x = 135° + 180° k$, where k is any integer.

These two general forms can be combined and expressed as $x = 45° + 90° k$, where k is any integer.

Section 2: Activity 1 (continued)

15. The graph of $y = \cos 4x$ is related to the graph of $y = \cos x$ by a horizontal compression factor of 4. In the interval $0 \le x < 360°$, $y = \cos x$ has one period; thus, $y = \cos 4x$ has four periods.

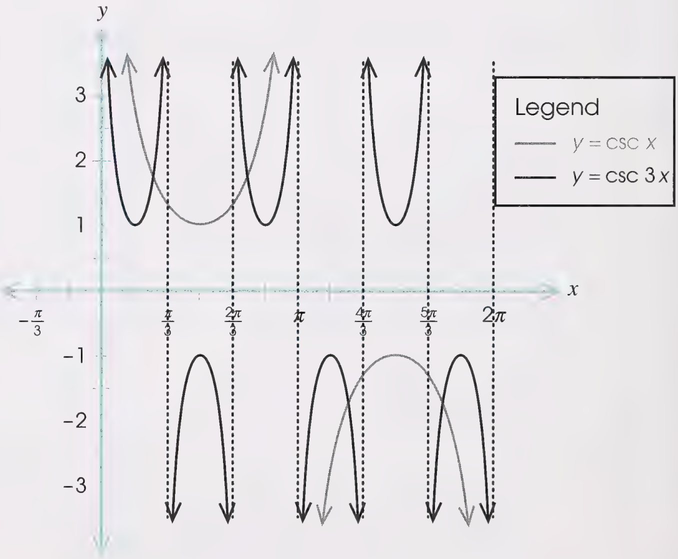


The period of $y = \cos 4x$ is $\frac{360°}{4}$ or 90°.

In the interval $0 \le x < 360°$, $\cos x = 1$ when $x = 0$. Therefore, $\cos 4x = 1$ when $x = 0$ in its first period.

Because the period of $y = \cos 4x$ is 90°, all the solutions of $\cos 4x = 1$ are $x = 90°k$, where k is any integer.

4. The graph of $y = \csc 3x$ is related to the graph of $y = \csc x$ by a horizontal compression factor of 3. In the interval $0 < x < 2\pi$, $y = \csc x$ has one period; thus, $y = \csc 3x$ has three periods.



The period of $y = \csc 3x$ is $\frac{2\pi}{3}$.

In the interval $0 < x < 2\pi$,

$\csc x = -\frac{2\sqrt{3}}{3}$ when $x = \frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

(**Note:** Use the special values from the unit circle for cosecant.) Therefore, in its first period, $\csc 3x = -\frac{2\sqrt{3}}{3}$ when

$$x = \frac{\frac{4\pi}{3}}{3} = \frac{4\pi}{9} \quad \text{and} \quad x = \frac{\frac{5\pi}{3}}{3} = \frac{5\pi}{9}$$

Recall that $y = \csc 3x$ has three periods in the interval $0 < x < 2\pi$. In the second period, $\csc x = -\frac{2\sqrt{3}}{3}$ when $x = \frac{4\pi}{9} + \frac{2\pi}{3}$ or $\frac{10\pi}{9}$ and when $x = \frac{5\pi}{9} + \frac{2\pi}{3}$ or $\frac{11\pi}{9}$. In the third period, $\csc x = -\frac{2\sqrt{3}}{3}$ when $x = \frac{10\pi}{9} + \frac{2\pi}{3}$ or $\frac{16\pi}{9}$ and when $x = \frac{11\pi}{9} + \frac{2\pi}{3}$ or $\frac{17\pi}{9}$.

The solutions of $\csc 3x = -\frac{2\sqrt{3}}{3}$ in the interval $0 < x < 2\pi$ are $\frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}$, and $\frac{17\pi}{9}$.

5. a. Textbook questions 16, 18, and 20 of "Practice," p. 248

16. The graph shows that $\tan x = 1$ in the interval $0 \le x < 360°$ when $x = 45°$ and $225°$.

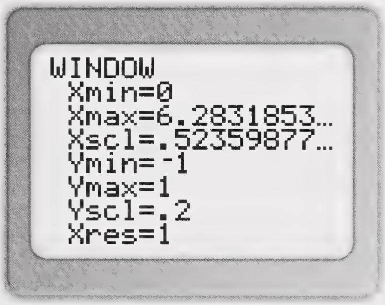
18. The graph of $y = \tan 3x$ is related to the graph of $y = \tan x$ by a horizontal compression factor of 3.

20. Because $y = \tan 3x$ is related to $y = \tan x$ by a horizontal compression factor of 3, the solutions to $\tan 3x = 1$ occur at $x = \frac{45°}{3}$ or 15° in its first period.

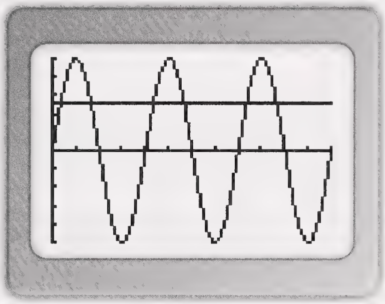
Because the period of $y = \tan 3x$ is $\frac{180°}{3}$ or 60°, all the solutions of $\tan 3x = 1$ are $x = 15° + 60°k$, where k is any integer.

b. Textbook questions 21, 24, 26, 27, and 29 of "Applications and Problem Solving," p. 248

21. a. Graph $y = \sin 3x$ and $y = \frac{1}{2}$ on your graphing calculator.



```
WINDOW
 Xmin=0
 Xmax=6.2831853…
 Xscl=.52359877…
 Ymin=-1
 Ymax=1
 Yscl=.2
 Xres=1
```



The solution to the equation $\sin 3x = \frac{1}{2}$ is given by the points of intersection of the curve $y = \sin 3x$ and the line $y = \frac{1}{2}$. In the interval $0 \le x < 2\pi$, points of intersection occur at $x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}$, and $\frac{29\pi}{18}$.

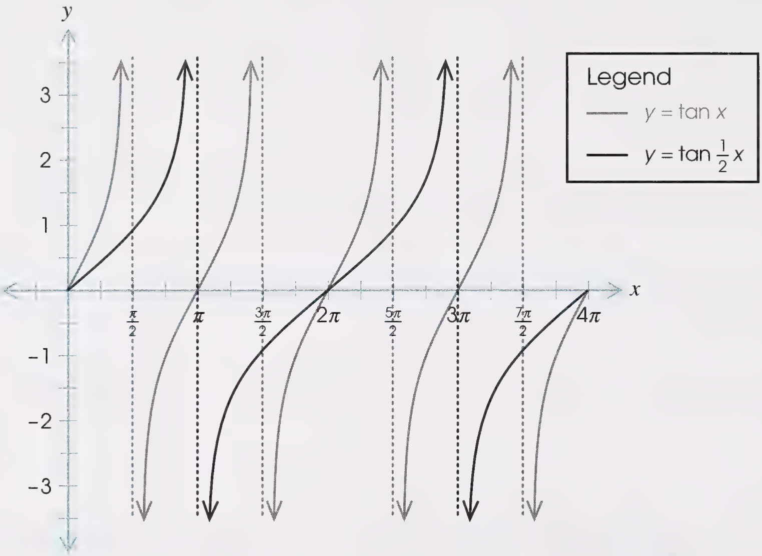
The solutions to the equation in the interval $0 \le x < 2\pi$ are $\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}$, and $\frac{29\pi}{18}$.

Section 2: Activity 1 (continued)

b. All coterminal angles will have the same sine value and differ by multiples of the period of the function. The period of $y = \sin x$ is 2π; so, the period of $y = \sin 3x$ is $\frac{2\pi}{3}$.

Because the solutions to $\sin 3x = \frac{1}{2}$ in one period are $x = \frac{\pi}{18}$ and $x = \frac{5\pi}{18}$, the general solutions are $x = \frac{\pi}{18} + \frac{2\pi}{3}k$ and $x = \frac{5\pi}{18} + \frac{2\pi}{3}k$, where k is any integer.

24. a.



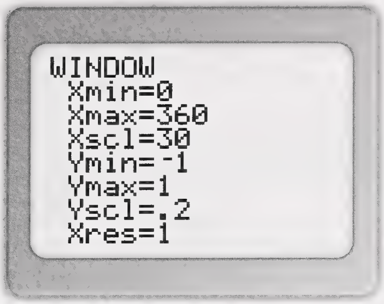
b. In the interval $0 \le x < 4\pi$, the graph of $y = \tan \frac{1}{2}x$ crosses the x-axis at $x = 0$ and $x = 2\pi$.

c. In the interval $0 \le x < 4\pi$, the graph of $y = \tan \frac{1}{4}x$ crosses the x-axis at one point only, $x = 0$. This is because the graph of $y = \frac{1}{4}x$ is related to the graph of $y = \tan x$ by a horizontal expansion of factor 4.

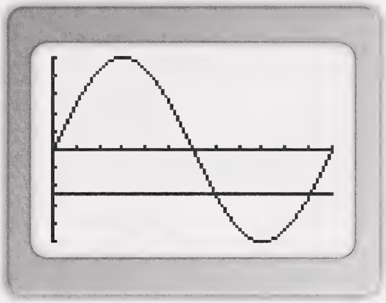
26. Solve for $\sin\theta$.

$$2\sin\theta + 1 = 0$$
$$2\sin\theta = -1$$
$$\sin\theta = -\frac{1}{2}$$

Graph $y = \sin\theta$ and $y = -\frac{1}{2}$ on your graphing calculator.



```
WINDOW
 Xmin=0
 Xmax=360
 Xscl=30
 Ymin=-1
 Ymax=1
 Yscl=.2
 Xres=1
```



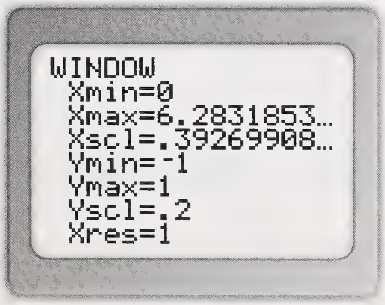
The graph shows that $\sin x = -\frac{1}{2}$ in the interval $0 \le x < 360°$ when $x = 210°$ and 330°.

The general solution is found by adding or subtracting multiples of the period of $y = \sin\theta$, which is 360°. Therefore, the general solutions for $2\sin\theta + 1 = 0$ are $x = 210° + 360°k$ and $x = 330° + 360°k$, where k is any integer.

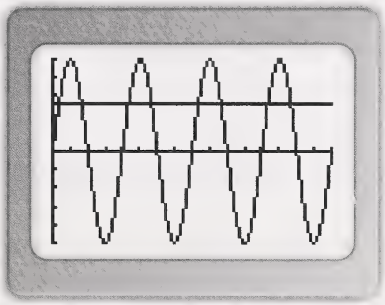
27. Solve for $\sin 4A$.

$$2\sin 4A - 1 = 0$$
$$2\sin 4A = 1$$
$$\sin 4A = \frac{1}{2}$$

Graph $y = \sin 4A$ and $y = \frac{1}{2}$ on your graphing calculator.



```
WINDOW
 Xmin=0
 Xmax=6.2831853…
 Xscl=.39269908…
 Ymin=-1
 Ymax=1
 Yscl=.2
 Xres=1
```



The period of $y = \sin 4A$ is $\frac{2\pi}{4}$ or $\frac{\pi}{2}$.

Section 2: Activity 1 (continued)

In the interval $0 \le x \le 2\pi$, $\sin A = \frac{1}{2}$ when $x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$. Therefore, in its first period, $\sin 4A = \frac{1}{2}$ when

$$x = \frac{\frac{\pi}{6}}{4} \quad \text{and} \quad x = \frac{\frac{5\pi}{6}}{4}$$

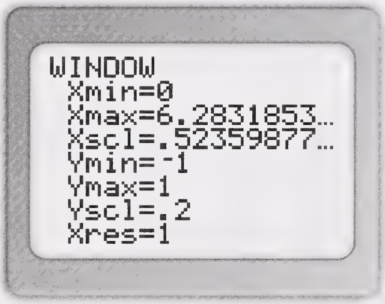
$$= \frac{\pi}{24} \qquad\qquad = \frac{5\pi}{24}$$

The period of $\sin 4A$ is $\frac{2\pi}{4}$ or $\frac{\pi}{2}$.

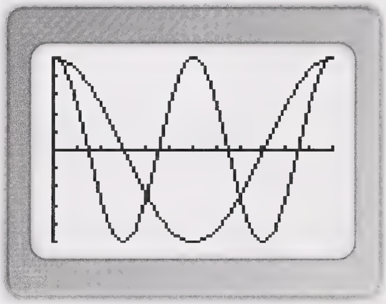
The graph of $y = \sin 4A$ has four periods in the interval $0 \le x \le 2\pi$. In the second period, $\sin 4A = \frac{1}{2}$ when $x = \frac{\pi}{24} + \frac{\pi}{2}$ or $\frac{13\pi}{24}$ and when $x = \frac{5\pi}{24} + \frac{\pi}{2}$ or $\frac{17\pi}{24}$. In the third period, $\sin 4A = \frac{1}{2}$ when $x = \frac{13\pi}{24} + \frac{\pi}{2}$ or $\frac{25\pi}{24}$ and when $x = \frac{17\pi}{24} + \frac{\pi}{2}$ or $\frac{29\pi}{24}$. In the fourth period, $\sin 4A = \frac{1}{2}$ when $x = \frac{25\pi}{24} + \frac{\pi}{2}$ or $\frac{37\pi}{24}$ and when $x = \frac{29\pi}{24} + \frac{\pi}{2}$ or $\frac{41\pi}{24}$.

The exact values that satisfy $\sin 4A = \frac{1}{2}$ are $\frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{25\pi}{24}, \frac{29\pi}{24}, \frac{37\pi}{24}$, and $\frac{41\pi}{24}$.

29. **a.** Graph $y = \cos 2x$ and $y = \cos x$ on your graphing calculator.



```
WINDOW
 Xmin=0
 Xmax=6.2831853…
 Xscl=.52359877…
 Ymin=-1
 Ymax=1
 Yscl=.2
 Xres=1
```

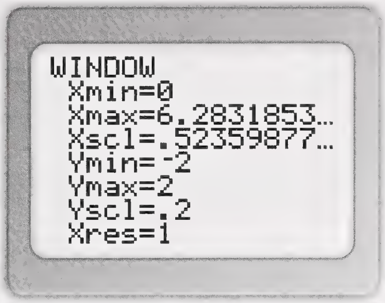


Find the points of intersection of the two graphs.

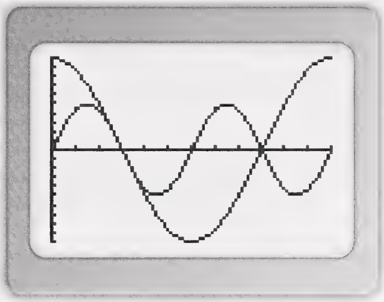
The graphs show that the solutions to $\cos 2x = \cos x$ in the interval $0 \le x < 2\pi$ are 0, $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$.

The remaining solutions are found in other periods of the function. These can be determined by adding or subtracting multiples of the period, 2π. Therefore, the general solutions for $\cos 2x = \cos x$ are $x = 0 + 2k\pi$, $x = \frac{2\pi}{3} + 2k\pi$, and $x = \frac{4\pi}{3} + 2k\pi$, where k is any integer.

b. Graph $y = \sin 2x$ and $y = 2\cos x$ on your graphing calculator.



```
WINDOW
 Xmin=0
 Xmax=6.2831853…
 Xscl=.52359877…
 Ymin=-2
 Ymax=2
 Yscl=.2
 Xres=1
```



Find the points of intersection of the two graphs.

The graphs show that the solutions to $\sin 2x = 2\cos x$ in the interval $0 \le x < 2\pi$ are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

The remaining solutions are found in other periods of the function. These can be determined by adding or subtracting multiples of the period, 2π. Therefore, the general solutions for $\sin 2x = 2\cos x$ are $x = \frac{\pi}{2} + 2k\pi$ and $x = \frac{3\pi}{2} + 2k\pi$, where k is any integer. These two general forms can be combined and expressed as $x = \frac{\pi}{2} + k\pi$ or $(2k+1)\frac{\pi}{2}$, where k is any integer.

6. You can solve this problem by trial and error.

$$\begin{array}{r} \text{FIVE} \\ \text{FIVE} \\ \text{NINE} \\ \underline{\text{ELEVEN}} \\ \text{THIRTY} \end{array} \quad \rightarrow \quad \begin{array}{r} 4027 \\ 4027 \\ 5057 \\ \underline{797275} \\ 810386 \end{array}$$

Therefore, $F = 4, I = 0, V = 2, E = 7, N = 5, L = 9, T = 8, H = 1, R = 3,$ and $Y = 6$.

Another approach is to use the power of technology. From the Pure Mathematics 30 Companion CD, open the file titled *fivesum.xls* from the Module_Work folder. The program will determine the solution. If you are interested in the method used to find the solution, use the Visual Basic Editor in *Excel*® to look at the program.

Section 2: Activity 2

1. a. **Textbook question "Explore: Use a Graph," p. 249**

The graphs appear to intersect at $x = -60°$ and $x = 60°$.

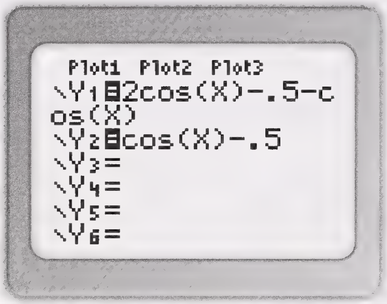
b. **Textbook questions 1 to 5 of "Inquire," p. 249**

1. The estimates are solutions to the equation $2\cos x = \frac{1}{2} + \cos x$.

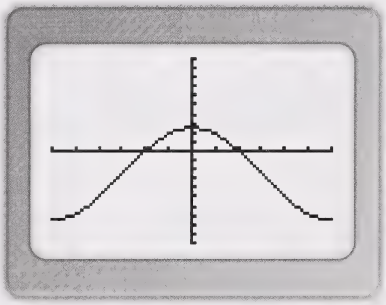
2. $$\begin{aligned} 2\cos x - 0.5 - \cos x &= 0 \\ 2\cos x &= 0.5 + \cos x \\ 2\cos x &= \frac{1}{2} + \cos x \end{aligned}$$

You can use the points where the graphs of $f(x) = 2\cos x$ and $g(x) = \frac{1}{2} + \cos x$ intersect to find the values of x. Therefore, the solutions to $2\cos x - 0.5 - \cos x = 0$ in the interval $-180° \leq x \leq 180°$ are $-60°$ and $60°$.

3. The graphs of $y = 2\cos x - \frac{1}{2} - \cos x$ and $y = \cos x - \frac{1}{2}$ will be identical.

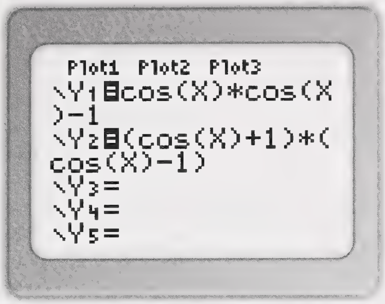


```
Plot1 Plot2 Plot3
\Y1=2cos(X)-.5-c
os(X)
\Y2=cos(X)-.5
\Y3=
\Y4=
\Y5=
\Y6=
```

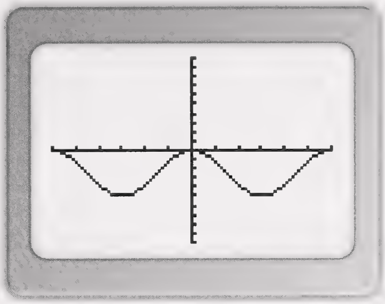


4. The solution for $\cos x - 1 = 0$ is the same as the solution for $2\cos x - 1 - \cos x = 0$. The first of these two equations is an algebraic simplification of the second, thus making the solutions the same.

5. The zeros of $f(x) = \cos^2 x - 1$ and $g(x) = (\cos x + 1)(\cos x - 1)$ will be the same. The second function is just a factored form of the first function.



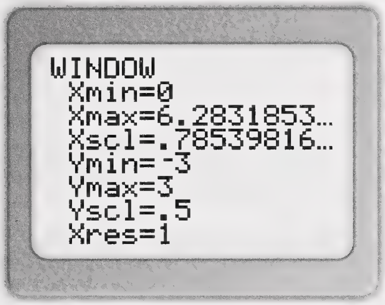
```
Plot1 Plot2 Plot3
\Y1■cos(X)*cos(X
)-1
\Y2■(cos(X)+1)*(
cos(X)-1)
\Y3=
\Y4=
\Y5=
```



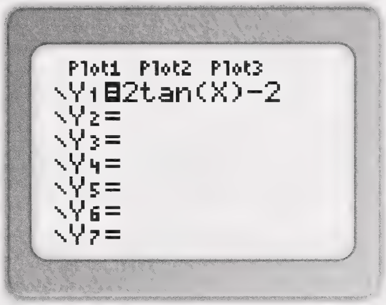
2. **Textbook questions 3, 4, 6, 10, 11, 12, 17, 18, 19, and 20 of "Practice," p. 252**

3. **Method 1: Solving Graphically**

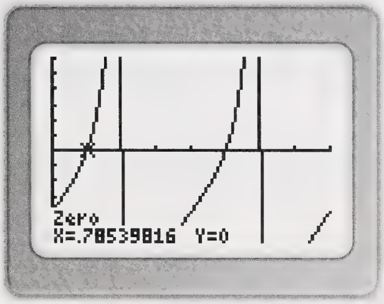
Use the Zero feature to determine the solutions to the equation.



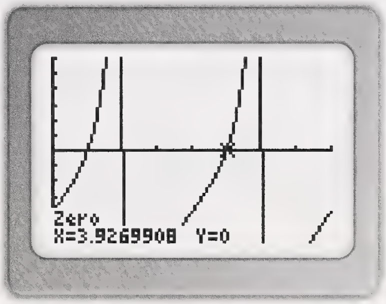
```
WINDOW
 Xmin=0
 Xmax=6.2831853…
 Xscl=.78539816…
 Ymin=-3
 Ymax=3
 Yscl=.5
 Xres=1
```



```
Plot1 Plot2 Plot3
\Y1■2tan(X)-2
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



```
Zero
X=.78539816  Y=0
```



```
Zero
X=3.9269908  Y=0
```

Because $\frac{\pi}{4} \doteq 0.7854$ and $\frac{5\pi}{4} \doteq 3.9270$, the solutions to $2\tan\theta - 2 = 0$ in the interval $0 \leq \theta < 2\pi$ are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

Section 2: Activity 2 (continued)

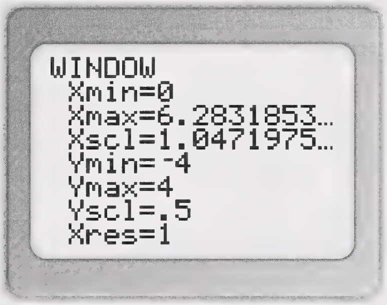
Method 2: Solving Algebraically

$$\begin{aligned} 2\tan\theta - 2 &= 0 \\ 2\tan\theta &= 2 \\ \tan\theta &= 1 \\ \theta &= \frac{\pi}{4} \end{aligned}$$

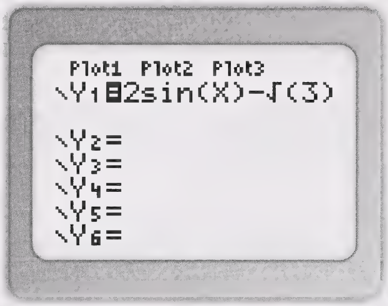
Because tangent is positive in the first and third quadrants, $\theta = \frac{\pi}{4}$ and $\frac{5\pi}{4}$.

4. **Method 1: Solving Graphically**

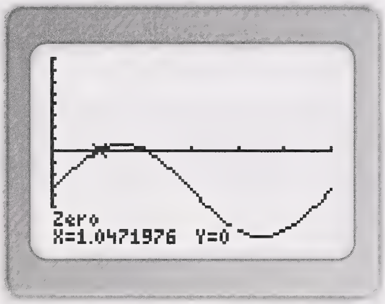
Use the Zero feature to determine the solutions to the equation.



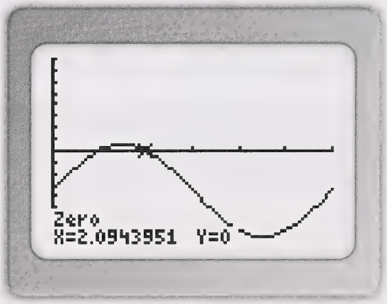
```
WINDOW
 Xmin=0
 Xmax=6.2831853…
 Xscl=1.0471975…
 Ymin=-4
 Ymax=4
 Yscl=.5
 Xres=1
```



```
 Plot1 Plot2 Plot3
\Y1=2sin(X)-√(3)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



```
Zero
X=1.0471976  Y=0
```



```
Zero
X=2.0943951  Y=0
```

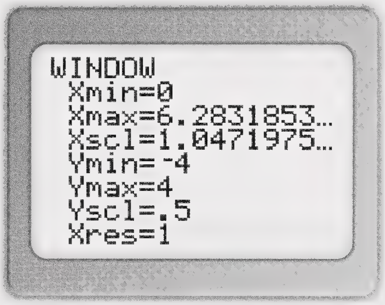
Because $\frac{\pi}{3} \doteq 1.0472$ and $\frac{2\pi}{3} \doteq 2.0944$, the solutions to $2\sin\theta - \sqrt{3} = 0$ in the interval $0 \le \theta < 2\pi$ are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

Method 2: Solving Algebraically

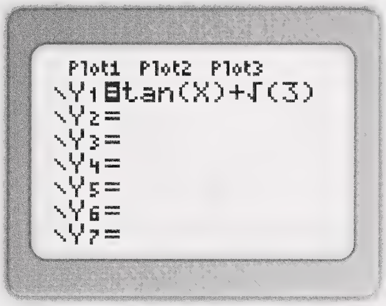
$$2\sin\theta-\sqrt{3}=0$$
$$2\sin\theta=\sqrt{3}$$
$$\sin\theta=\frac{\sqrt{3}}{2}$$
$$\theta=\frac{\pi}{3}$$

Because sine is positive in the first and second quadrants, $\theta=\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

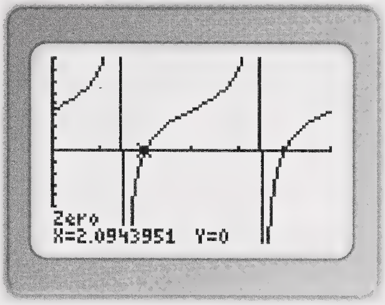
6. Method 1: Solving Graphically



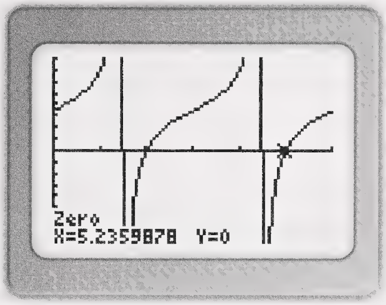
```
WINDOW
  Xmin=0
  Xmax=6.2831853…
  Xscl=1.0471975…
  Ymin=-4
  Ymax=4
  Yscl=.5
  Xres=1
```



```
 Plot1 Plot2 Plot3
\Y1■tan(X)+√(3)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



```
Zero
X=2.0943951   Y=0
```



```
Zero
X=5.2359878  Y=0
```

Because $\frac{2\pi}{3}\doteq 2.0944$ and $\frac{5\pi}{3}\doteq 5.2360$, the solutions to $\tan\theta+\sqrt{3}=0$ in the interval $0\le\theta<2\pi$ are $\theta=\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

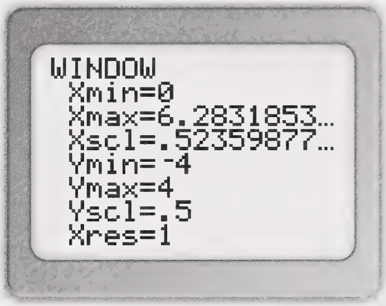
Section 2: Activity 2 (continued)

Method 2: Solving Algebraically

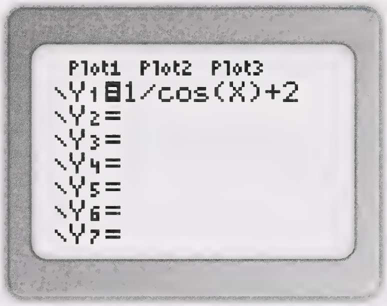
$$\tan\theta + \sqrt{3} = 0$$
$$\tan\theta = -\sqrt{3}$$
$$\theta = \frac{2\pi}{3}$$

Because tangent is negative in the second and fourth quadrants, $\theta = \frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

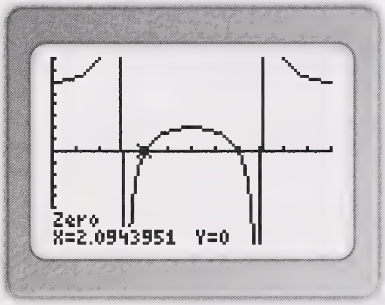
10. Method 1: Solving Graphically



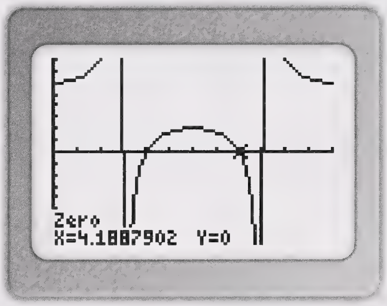
```
WINDOW
 Xmin=0
 Xmax=6.2831853…
 Xscl=.52359877…
 Ymin=-4
 Ymax=4
 Yscl=.5
 Xres=1
```



```
 Plot1 Plot2 Plot3
\Y1=1/cos(X)+2
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



```
Zero
X=2.0943951  Y=0
```



```
Zero
X=4.1887902  Y=0
```

Because $\frac{2\pi}{3} \doteq 2.0944$ and $\frac{4\pi}{3} \doteq 4.1888$, the solutions to $\sec\theta = -2$ in the interval $0 \le \theta < 2\pi$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Method 2: Solving Algebraically

$$\sec\theta = -2$$

$$\frac{1}{\cos\theta} = -2$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}$$

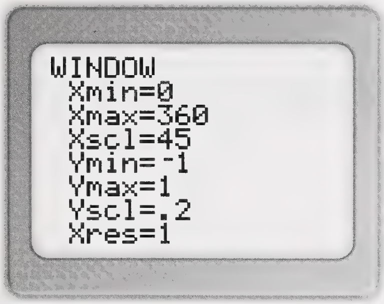
Because cosine is negative in the second and third quadrants, $\theta = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

11. The graph of $y = \cos 2x$ is related to the graph of $y = \cos x$ by a horizontal compression factor of 2. Because $\cos x = 0$ at $x = 90°$ and $x = 270°$ in the interval $0 \le x < 360°$, $\cos 2x = 0$ has zeros at $x = \frac{90°}{2}$ or 45° and $x = \frac{270°}{2}$ or 135° in its first period.

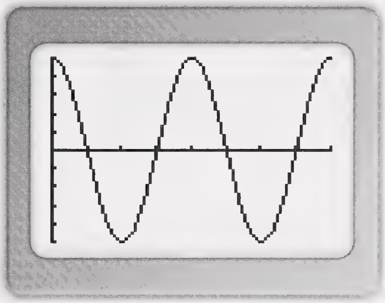
The graph of $y = \cos 2x$ has two periods in the interval $0 \le x < 360°$. In the second period, $\cos 2x = 0$ when $x = 45° + 180°$ or 225° and when $x = 135° + 180°$ or 315°.

Therefore, the solutions to $\cos 2x = 0$ in the interval $0 \le x \le 360°$ are 45°, 135°, 225°, and 315°.

Check



```
WINDOW
  Xmin=0
  Xmax=360
  Xscl=45
  Ymin=-1
  Ymax=1
  Yscl=.2
  Xres=1
```



Section 2: Activity 2 (continued)

17. $2 \sin 2x = 1$

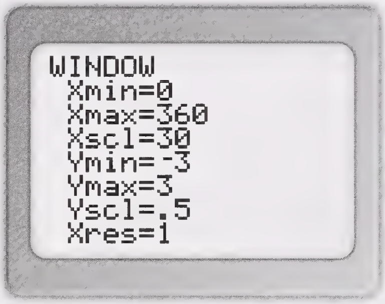
$$\sin 2x = \frac{1}{2}$$

The graph of $y = \sin 2x$ is related to the graph of $y = \sin x$ by a horizontal compression factor of 2. Because $\sin x = \frac{1}{2}$ has the solutions $x = 30°$ and $x = 150°$ in the interval $0 \le x < 360°$, $\sin 2x = \frac{1}{2}$ has solutions at $x = \frac{30°}{2}$ or 15° and $x = \frac{150°}{2}$ or 75° in its first period.

The graph of $y = \sin 2x$ has two periods in the interval $0 \le x < 360°$. In the second period, $\sin 2x = \frac{1}{2}$ when $x = 15° + 180°$ or 195° and when $x = 75° + 180°$ or 255°.

Therefore, the solutions for $2 \sin 2x = 1$ in the interval $0 \le x \le 360°$ are 15°, 75°, 195°, and 225°.

Check



```
WINDOW
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  Xmax=360
  Xscl=30
  Ymin=-3
  Ymax=3
  Yscl=.5
  Xres=1
```
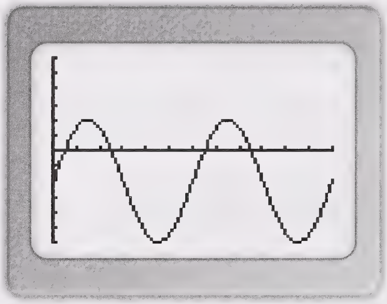
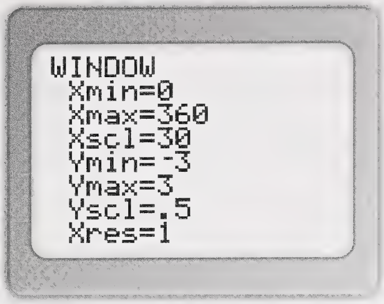



18. $2 \cos \frac{1}{2} x = 1$

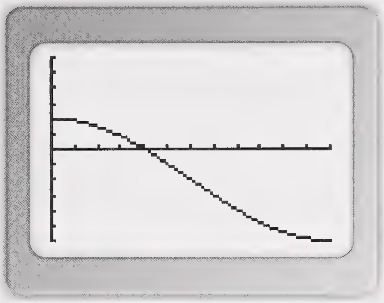
$$\cos \frac{1}{2} x = \frac{1}{2}$$

The graph of $\cos \frac{1}{2} x$ is related to the graph of $y = \cos x$ by a horizontal expansion factor of 2. Since $\cos x = \frac{1}{2}$ has solutions of $x = 60°$ and $x = 300°$ in the interval $0 \le x \le 360°$, $\cos \frac{1}{2} x = \frac{1}{2}$ will have solutions at $x = 60° \times 2$ or 120° and $x = 300° \times 2$ or 600°. Because the second value is outside the given domain, the solution to $2 \cos \frac{1}{2} x = 1$ is 120°.

Check



```
WINDOW
 Xmin=0
 Xmax=360
 Xscl=30
 Ymin=-3
 Ymax=3
 Yscl=.5
 Xres=1
```



19. $2 \cos 3x = 1$

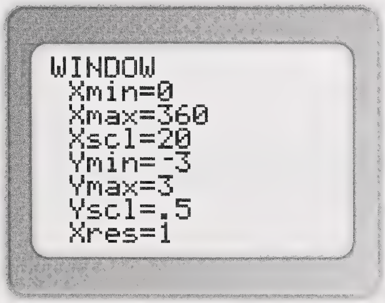
$$\cos 3x = \frac{1}{2}$$

The graph of $y = \cos 3x$ is related to the graph of $y = \cos x$ by a horizontal compression factor of 3. Because $\cos x = \frac{1}{2}$ has the solutions $x = 60°$ and $x = 300°$ in the interval $0 \le x \le 360°$, $\cos 3x = \frac{1}{2}$ will have solutions at $x = \frac{60°}{3}$ or 20° and $x = \frac{300°}{3}$ or 100°.

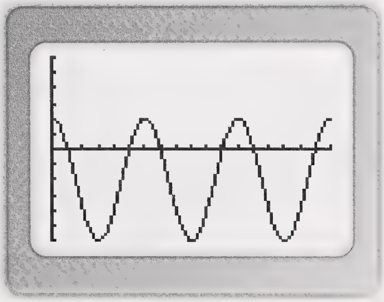
The graph of $y = \cos 3x$ has three periods of 120° in the interval $0 \le x < 360°$. In the second period, $\cos 3x = \frac{1}{2}$ has solutions at $x = 20° + 120°$ or 140° and at $x = 100° + 120°$ or 220°. In the third period, $\cos 3x = \frac{1}{2}$ has solutions at $x = 140° + 120°$ or 260° and $x = 220° + 120°$ or 340°.

Therefore, the solutions to $2 \cos 3x = 1$ in the interval $0 \le x < 360°$ are 20°, 100°, 140°, 220°, 260°, and 340°.

Check



```
WINDOW
 Xmin=0
 Xmax=360
 Xscl=20
 Ymin=-3
 Ymax=3
 Yscl=.5
 Xres=1
```

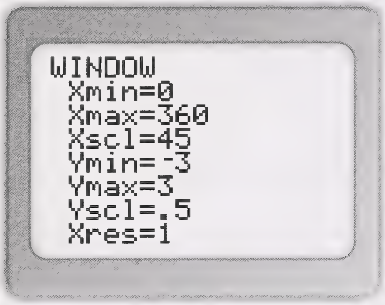


Section 2: Activity 2 (continued)

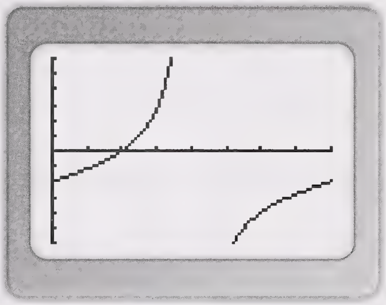
20. The graph of $y = \tan \frac{1}{2}x$ is related to the graph of $y = \tan x$ by a horizontal expansion factor of 2.

Because $\tan x = 1$ has solutions of $x = 45°$ and $x = 225°$ in the interval $0 \le x \le 360°$, $\tan \frac{1}{2}x = 1$ will have solutions at $x = 45° \times 2$ or 90° and $x = 225° \times 2$ or 450°. Because the second value is outside the given domain, the solution to $\tan \frac{1}{2}x = 1$ in the interval $0 \le x < 360°$ is 90°.

Check



```
WINDOW
 Xmin=0
 Xmax=360
 Xscl=45
 Ymin=-3
 Ymax=3
 Yscl=.5
 Xres=1
```



3. Textbook questions 21 and 22 of "Practice," p. 252

21.

$$\cos^2 x = 1$$
$$\cos^2 x - 1 = 0$$
$$(\cos x - 1)(\cos x + 1) = 0$$

In the interval $0 \le \theta < 2\pi$,

$$\cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0$$
$$\cos x = 1 \qquad\qquad \cos x = -1$$
$$x = 0 \qquad\qquad x = \pi$$

The general solution is found by adding multiples of the period of the function, 2π, to the solutions. Therefore, the general solutions for $\cos^2 x = 1$ are $x = 2k\pi$ and $x = \pi + 2k\pi$, where *k* is any integer. These two equations can be combined as $x = k\pi$, where *k* is any integer.

22. $$\sin^2 x - 1 = 0$$
$$(\sin x - 1)(\sin x + 1) = 0$$

In the interval $0 \le x < 2\pi$,

$$\begin{aligned} \sin x - 1 &= 0 &\quad \text{or} \quad&& \sin x + 1 &= 0 \\ \sin x &= 1 &&& \sin x &= -1 \\ x &= \frac{\pi}{2} &&& x &= \frac{3\pi}{2} \end{aligned}$$

The general solution is found by adding multiples of the period of the function, 2π, to the solutions. Therefore, the general solutions for $\sin^2 x - 1 = 0$ are $x = \frac{\pi}{2} + 2k\pi$ and $x = \frac{3\pi}{2} + 2k\pi$, where k is any integer. These two equations can be combined as $x = \frac{\pi}{2} + k\pi$ or $(2k+1)\frac{\pi}{2}$, where k is any integer.

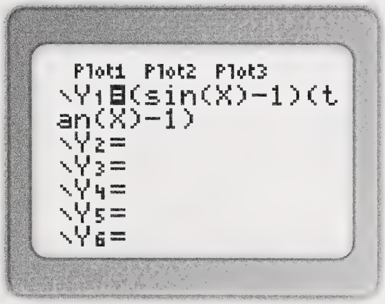
4. a. Textbook questions 25, 29, and 30 of "Practice," p. 252

25. $(\sin x - 1)(\tan x - 1) = 0$

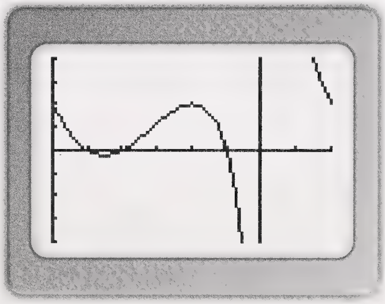
$$\begin{aligned} \therefore\ \sin x - 1 &= 0 &\quad \text{or} \quad&& \tan x - 1 &= 0 \\ \sin x &= 1 &&& \tan x &= 1 \\ x &= \frac{\pi}{2} &&& x &= \frac{\pi}{4} \text{ and } \frac{5\pi}{4} \end{aligned}$$

The solutions to $(\sin x - 1)(\tan x - 1) = 0$ in the interval $0 \le x < 2\pi$ are $\frac{\pi}{4}$, $\frac{\pi}{2}$, and $\frac{5\pi}{4}$.

Check



```
Plot1 Plot2 Plot3
\Y1=(sin(X)-1)(t
an(X)-1)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



The period of $\sin x$ is 2π. To account for all coterminal angles, $x = \frac{\pi}{2} + 2n\pi$, where n is any integer.

Section 2: Activity 2 (continued)

The period of $\tan x$ is π. To account for all coterminal angles, $x=\frac{\pi}{4}+n\pi$ and $x=\frac{5\pi}{4}+n\pi$, where *n* is any integer. These two general forms can be combined as $x=\frac{\pi}{4}+n\pi$, where *n* is any integer.

***Remember:** The general solution is found by adding multiples of the period.*

Therefore, the general solutions for $(\sin x-1)(\tan x-1)=0$ are $x=\frac{\pi}{2}+2n\pi$ and $x=\frac{\pi}{4}+n\pi$, where *n* is any integer.

29. $\cos^2 2x+\cos 2x=0$

$$\cos 2x(\cos 2x+1)=0$$

$$\therefore\ \cos 2x=0 \qquad \text{or} \qquad \cos 2x+1=0$$

$$2x=\frac{\pi}{2}\text{ and }\frac{3\pi}{2} \qquad\qquad \cos 2x=-1$$

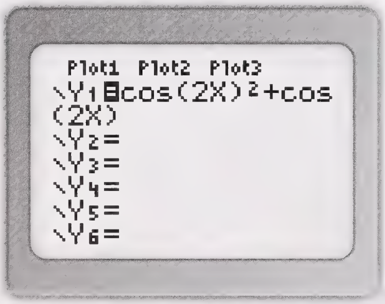
$$2x=\pi$$

$$x=\frac{\pi}{4}\text{ and }\frac{3\pi}{4} \qquad\qquad x=\frac{\pi}{2}$$

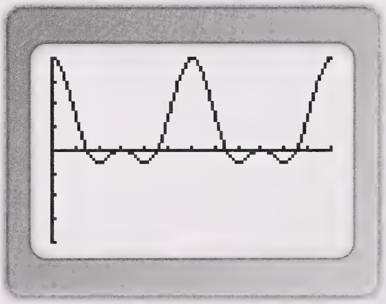
Because $\cos 2x$ has two periods in the interval $0\le x<2\pi$, the solutions in the second period are $x=\frac{\pi}{4}+\pi$ or $\frac{5\pi}{4}$, $x=\frac{3\pi}{4}+\pi$ or $\frac{7\pi}{4}$, and $x=\frac{\pi}{2}+\pi$ or $\frac{3\pi}{2}$.

Therefore, the solutions to $\cos^2 2x+\cos 2x=0$ in the interval $0\le x<2\pi$ are $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$, and $\frac{7\pi}{4}$.

Check



```
  Plot1  Plot2  Plot3
\Y1■cos(2X)²+cos
(2X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



The period of $\cos 2x$ is π. To account for all coterminal angles, $x=\frac{\pi}{4}+n\pi$, $x=\frac{\pi}{2}+n\pi$, and $x=\frac{3\pi}{4}+n\pi$, where *n* is any integer. These are the general solutions.

30.

$$\begin{aligned} 4\sin^2 x + 2\sin x - 2 &= 0 \\ 2(2\sin^2 x + \sin x - 1) &= 0 \\ 2(2\sin x - 1)(\sin x + 1) &= 0 \\ (2\sin x - 1)(\sin x + 1) &= 0 \end{aligned}$$

$$\therefore\ 2\sin x - 1 = 0 \qquad \text{or} \qquad \sin x + 1 = 0$$

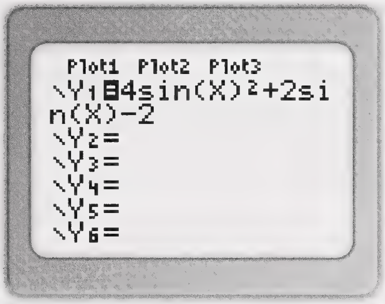
$$2\sin x = 1 \qquad\qquad \sin x = -1$$

$$\sin x = \frac{1}{2} \qquad\qquad x = \frac{3\pi}{2}$$

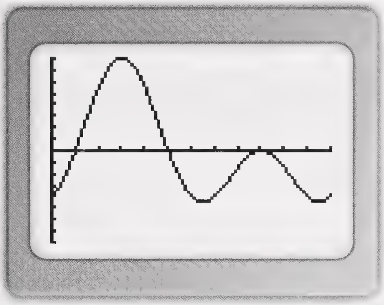
$$x = \frac{\pi}{6} \text{ and } \frac{5\pi}{6}$$

The solutions to $4\sin^2 x + 2\sin x - 2 = 0$ in the interval $0 \le x < 2\pi$ are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, and $\frac{3\pi}{2}$.

Check



```
Plot1 Plot2 Plot3
\Y1■4sin(X)²+2si
n(X)-2
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



The period of $\sin x$ is 2π. To account for all coterminal angles, $x = \frac{\pi}{6} + 2n\pi$, $x = \frac{5\pi}{6} + 2n\pi$, and $x = \frac{3\pi}{2} + 2n\pi$, where n is any integer. These are the general solutions.

Section 2: Activity 2 (continued)

b. Textbook questions 33, 35, 36, and 40 of "Applications and Problem Solving," p. 253

33. a.

$$\begin{aligned} 3\tan^2 x + \tan x &= 4 \\ 3\tan^2 x + \tan x - 4 &= 0 \\ (3\tan x + 4)(\tan x - 1) &= 0 \end{aligned}$$

$$\therefore\ 3\tan x + 4 = 0 \qquad \text{or} \qquad \tan x - 1 = 0$$

$$3\tan x = -4 \qquad\qquad \tan x = 1$$

$$\tan x = -\frac{4}{3} \qquad\qquad x = \frac{\pi}{4} \text{ and } \frac{5\pi}{4}$$

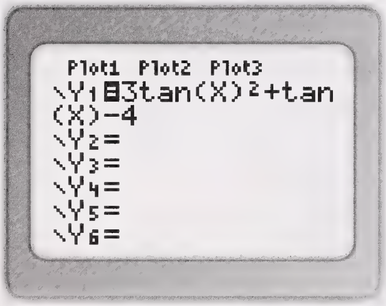
$$x \doteq -0.9273$$

But $x = -0.9273$ is not in the domain $0 \le x < 2\pi$. Because the period of tan x is π, to find solutions in $0 \le x < 2\pi$, add π and 2π to -0.9273.

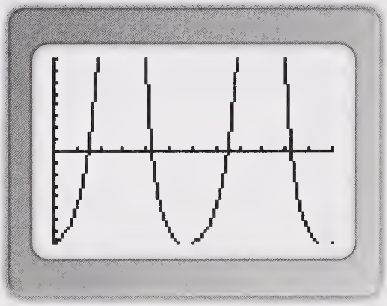
$$\begin{aligned} x &\doteq -0.9273 + \pi \\ &\doteq 2.2143 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= -0.9273 + 2\pi \\ &= 5.3559 \end{aligned}$$

The solutions to $3\tan^2 x + \tan x = 4$ in the interval $0 \le x < 2\pi$ are $\frac{\pi}{4}$, 2.2143, $\frac{5\pi}{4}$, and 5.3559.

Check



```
Plot1 Plot2 Plot3
\Y1■3tan(X)²+tan
(X)-4
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



b. $\sin^2 x + \sin x - 2 = 0$

$(\sin x + 2)(\sin x - 1) = 0$

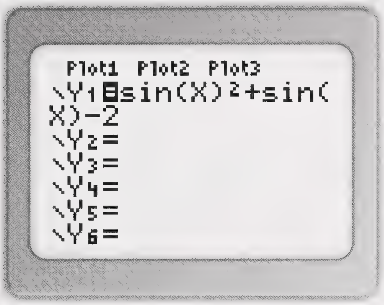
$\therefore \sin x + 2 = 0$ or $\sin x - 1 = 0$

$\sin x = -2$ $\qquad$ $\sin x = 1$

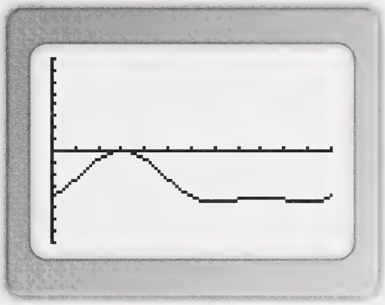
$x = \text{undefined}$ $\qquad$ $x = \frac{\pi}{2}$

Therefore, the solution to $\sin^2 x + \sin x - 2 = 0$ in the interval $0 \le x < 2\pi$ is $\frac{\pi}{2}$.

Check



```
 Plot1  Plot2  Plot3
\Y1■sin(X)²+sin(
X)-2
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



c. $2\cos^2 x - 3\cos x + 1 = 0$

$(2\cos x - 1)(\cos x - 1) = 0$

$\therefore 2\cos x - 1 = 0$ or $\cos x - 1 = 0$

$2\cos x = 1$ $\qquad$ $\cos x = 1$

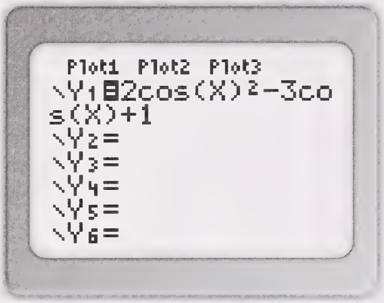
$\cos x = \frac{1}{2}$ $\qquad$ $x = 0$

$x = \frac{\pi}{3}$ and $\frac{5\pi}{3}$

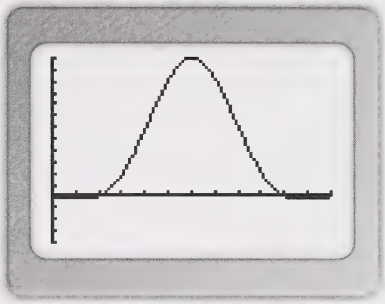
The solutions to $2\cos^2 x - 3\cos x + 1 = 0$ in the interval $0 \le x < 2\pi$ are 0, $\frac{\pi}{3}$, and $\frac{5\pi}{3}$.

Section 2: Activity 2 (continued)

Check



```
 Plot1 Plot2 Plot3
\Y1■2cos(X)²-3co
s(X)+1
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



35. **a.** $5\tan^2 x + 2\tan x - 7 = 0$

$(5\tan x + 7)(\tan x - 1) = 0$

$\therefore\ 5\tan x + 7 = 0$ or $\tan x - 1 = 0$

$5\tan x = -7$ $\qquad$ $\tan x = 1$

$\tan x = -\frac{7}{5}$ $\qquad$ $x = \frac{\pi}{4}$ and $\frac{5\pi}{4}$

$x \doteq 2.1910$ and 5.3326 $\qquad$ $\doteq 0.7854$ and 3.9270

The solutions to $5\tan^2 x + 2\tan x - 7 = 0$ in the interval $0 \le x < 2\pi$ are about 0.7854, 2.1910, 3.9270, and 5.3326.

b. $\tan^2 x - 5\tan x + 6 = 0$

$(\tan x - 3)(\tan x - 2) = 0$

$\therefore\ \tan x - 3 = 0$ or $\tan x - 2 = 0$

$\tan x = 3$ $\qquad$ $\tan x = 2$

$x \doteq 1.2490$ and 4.3906 $\qquad$ $x \doteq 1.1071$ and 4.2487

The solutions to $\tan^2 x - 4\tan x = 0$ in the interval $0 \le x < 2\pi$ are about 1.1071, 1.2490, 4.2487, and 4.3906.

c. $\tan^2 x - 4\tan x = 0$

$\tan x(\tan x - 4) = 0$

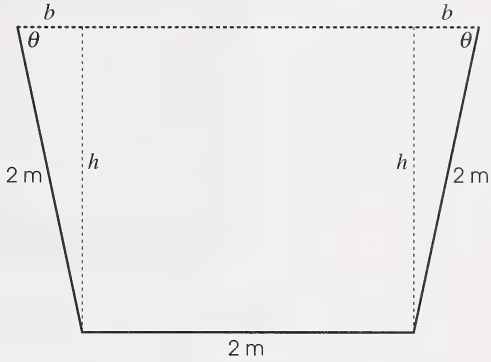
$\therefore\ \tan x = 0$ or $\tan x - 4 = 0$

$x = 0$ and π $\qquad$ $\tan x = 4$

$\doteq 0$ and 3.1416 $\qquad$ $x \doteq 1.3258$ and 4.4674

The solutions to $5\tan^2 x + 2\tan x - 7 = 0$ in the interval $0 \le x < 2\pi$ are about 0, 1.3258, 3.1416, and 4.4674.

36. a. The area of a trapezoid is found using its height and the lengths of its parallel sides.



$\therefore\ \sin\theta = \dfrac{h}{2}$ and $\cos\theta = \dfrac{b}{2}$

$h = 2\sin\theta$ $\qquad$ $b = 2\cos\theta$

The length of the shorter parallel side, b_1, is 2 m.

The length of the longer parallel side, b_2, is

$$\begin{aligned} 2 + 2b &= 2 + 2(2\cos\theta) \\ &= 2 + 4\cos\theta \end{aligned}$$

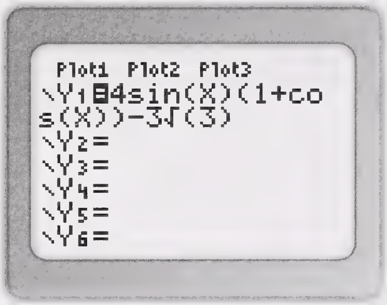
$$\begin{aligned} \therefore\ A &= \frac{(b_1 + b_2)h}{2} \\ &= \frac{[2 + (2 + 4\cos\theta)]2\sin\theta}{2} \\ &= (4 + 4\cos\theta)\sin\theta \\ &= 4\sin\theta + 4\sin\theta\cos\theta \\ &= 4\sin\theta(1 + \cos\theta) \end{aligned}$$

Section 2: Activity 2 (continued)

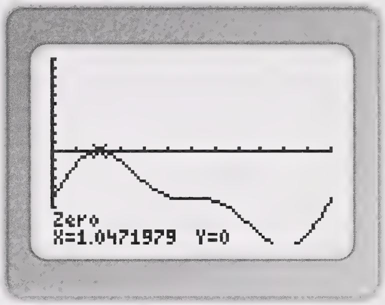
b.

$$\begin{aligned} A &= 4\sin\theta(1+\cos\theta) \\ 3\sqrt{3} &= 4\sin\theta(1+\cos\theta) \\ 0 &= 4\sin\theta(1+\cos\theta)-3\sqrt{3} \end{aligned}$$

Graph the equation on your graphing calculator, and use the Zero feature to determine the value of θ.



```
Plot1 Plot2 Plot3
\Y1■4sin(X)(1+co
s(X))-3√(3)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



```
Zero
X=1.0471979  Y=0
```

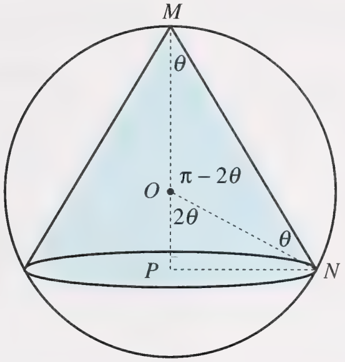
Because $\frac{\pi}{3} \doteq 1.0472$, according to the graph, the value of θ is $\frac{\pi}{3}$ or 60°.

Check

LS	RS
$3\sqrt{3}$	$4\sin\theta(1+\cos\theta)$
	$=4\sin\frac{\pi}{3}\left(1+\cos\frac{\pi}{3}\right)$
	$=4\left(\frac{\sqrt{3}}{2}\right)\left(1+\frac{1}{2}\right)$
	$=4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{3}{2}\right)$
	$=4\left(\frac{3\sqrt{3}}{4}\right)$
	$=3\sqrt{3}$

LS = RS

40. a. Notice that ΔMNO is an isosceles triangle; so, $\angle MNO = \angle NMO$. (In this case, they are both represented by θ.) That makes $\angle MON = \pi - 2\theta$ and $\angle PON = 2\theta$.



The volume of a circular cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, where h is the height of the cone and r is the radius of the base of the cone. To find the volume, the height of the cone and radius of the base of the cone are needed.

The height, h, of the cone is the sum of the lengths of MO and OP. MO and NO are radii of the circle; thus $MO = NO = 30$ cm and $OP = 30\cos 2\theta$.

$\cos 2\theta = \frac{OP}{30}$

$OP = 30\cos 2\theta$

$$\therefore\ h = 30 + 30\cos 2\theta$$
$$= 30(1 + \cos 2\theta)$$

The radius, r, of the cone is PN.

$$\therefore\ r = 30\sin 2\theta$$

$\sin 2\theta = \frac{PN}{30}$

$PN = 30\sin 2\theta$

Substitute $h = 30(1 + \cos 2\theta)$ and $r = 30\sin 2\theta$ into the formula for volume of a right-circular cone.

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(30\sin 2\theta)^2\left[30(1 + \cos 2\theta)\right]$$
$$= \frac{\pi(900\sin^2 2\theta)(\overset{10}{\cancel{30}})(1 + \cos 2\theta)}{\underset{1}{\cancel{3}}}$$
$$= 9000\pi\sin^2 2\theta(1 + \cos 2\theta)$$

Section 2: Activity 2 (continued)

b.

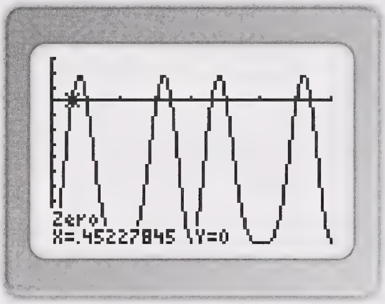
$$V = 9000\pi \sin^2 2\theta(1+\cos 2\theta)$$

$$\cancel{9000\pi} = \cancel{9000\pi} \sin^2 2\theta(1+\cos 2\theta)$$

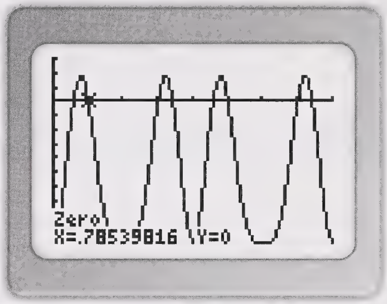
$$1 = \sin^2 2\theta(1+\cos 2\theta)$$

From the diagram, θ must be an acute angle! Therefore, $0 < \theta < 90°$ or $0 < \theta < \frac{\pi}{2}$. Notice that the equation will be true when $|\sin 2\theta| = 1$ and $\cos 2\theta = 0$. This will occur when $\theta = \frac{\pi}{4}$.

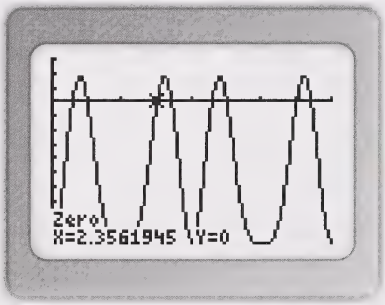
Use the Zero feature on your graphing calculator to find any other values for θ.



```
Zero
X=.45227845  Y=0
```



```
Zero
X=.78539816  Y=0
```



```
Zero
X=2.3561945  Y=0
```

This is too large.
$2.3562 \doteq 135°$

Therefore, the values of θ are 45° and approximately 25.9° or $\frac{\pi}{4}$ and approximately 0.4523.

5. The graph looks like the sine function, except instead of being a horizontal graph it appears to cross the x-axis at 45°. Recall that the equation of a 45° line is $y = x$. Try combinations of $y = x$ and $y = \sin x$ to see which matches the given graph.

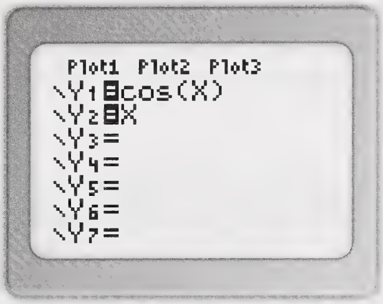
By trial and error, the equation of the graph is $y = x + \sin x$.

Section 2: Activity 3

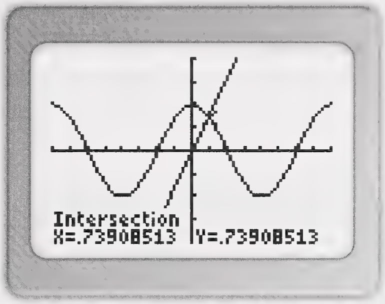
1. **a.** **Textbook questions a. and b. of "Explore: Use a Graph," p. 254**

a. The graphs seem to meet at $\frac{\pi}{4}$.

b.



```
Plot1 Plot2 Plot3
\Y1■cos(X)
\Y2■X
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



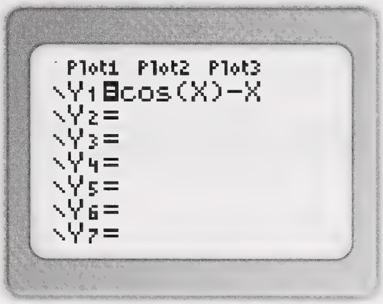
```
Intersection
X=.73908513  Y=.73908513
```

The calculator gives a value of 0.7391, correct to four decimal places.

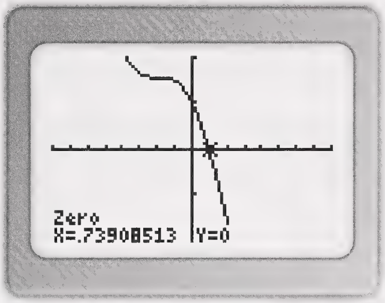
b. **Textbook questions 1 to 4 of "Inquire," p. 254**

1. The x-value at the point of intersection of $y = \cos x$ and $y = x$ is the zero of $y = \cos x - x$ or the solution to $\cos x - x = 0$.

2.



```
Plot1 Plot2 Plot3
\Y1■cos(X)-X
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



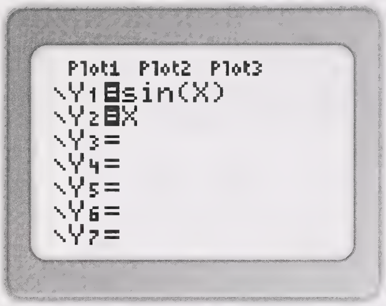
```
Zero
X=.73908513  Y=0
```

The solution to the equation is approximately 0.7391.

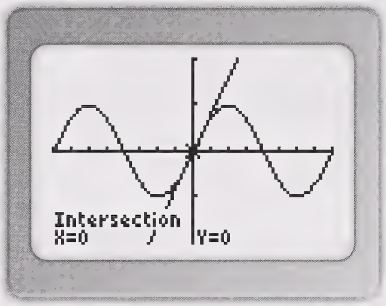
Section 2: Activity 3 (continued)

3. This is the only solution of $y = \cos x - x$. Cosine is never larger than 1 nor less than -1. The only domain for $y = x$ that matches this range is $-1 \le x \le 1$. The graph shows this domain and shows only one x-intercept. Over the domain $-1 \le x \le 0$, cosine is positive and, thus, is larger than x, which is negative. Over the domain $0 \le x \le 1$, cosine is decreasing and x is increasing; so, there will be at most one point of intersection.

4. Graph $y = \sin x$ and $y = x$ on your graphing calculator. The solution to $\sin x - x = 0$ is the value of x at the point of intersection of the graphs.

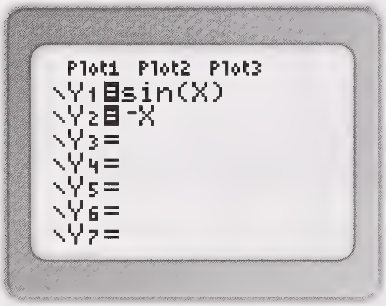


```
Plot1 Plot2 Plot3
\Y1=sin(X)
\Y2=X
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

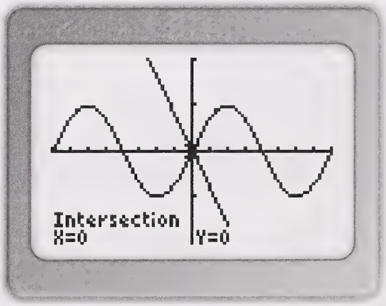


The only solution to $\sin x - x = 0$ is $x = 0$.

Graph $y = \sin x$ and $y = -x$. The solution of $\sin x + x = 0$ is the point of intersection of these graphs.



```
Plot1 Plot2 Plot3
\Y1=sin(X)
\Y2= -X
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



The only solution to $\sin x + x = 0$ is also 0.

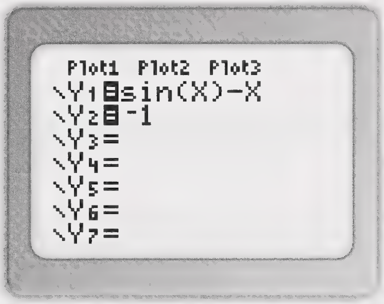
2. Textbook questions 1 to 7 of "Practice," pp. 256 and 257

1. The intersection of the graphs is the solution to the equation $\sin x = x - 1$. An approximate solution is $x = 1.9635$.

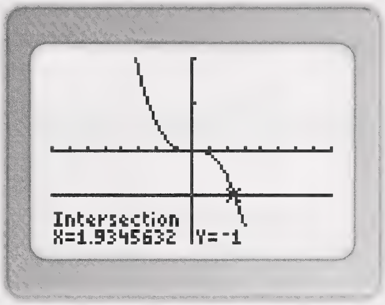
2. $\sin x = x - 1$

   $\sin x - x = -1$

   Graph $y = \sin x - x$ and $y = -1$ on your graphing calculator. The intersection of these graphs is the solution to $\sin x = x - 1$.

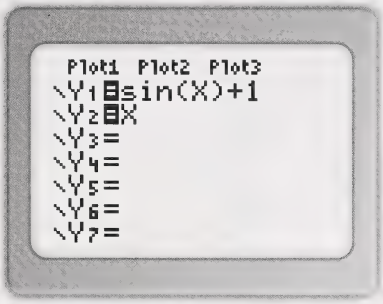


```
Plot1  Plot2  Plot3
\Y1=sin(X)-X
\Y2=-1
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

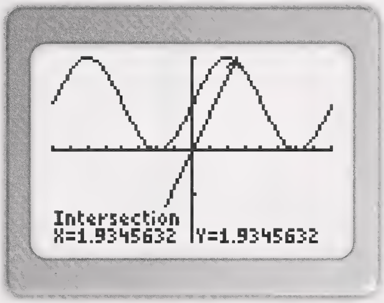


```
Intersection
X=1.9345632  Y=-1
```

3. The graphs of $y = \sin x + 1$ and $y = x$ would lead to the same intersection as found in questions 1 and 2. The point of intersection would be (1.9346, 1.9346), correct to four decimal places.



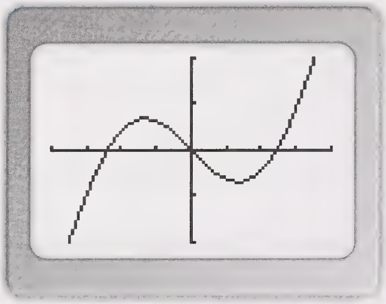
```
Plot1  Plot2  Plot3
\Y1=sin(X)+1
\Y2=X
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



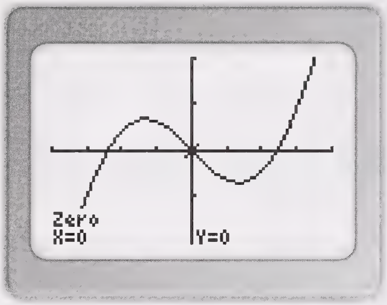
```
Intersection
X=1.9345632  Y=1.9345632
```

Section 2: Activity 3 (continued)

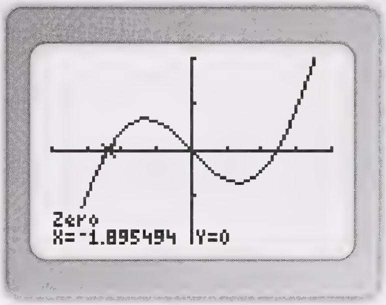
4. a.



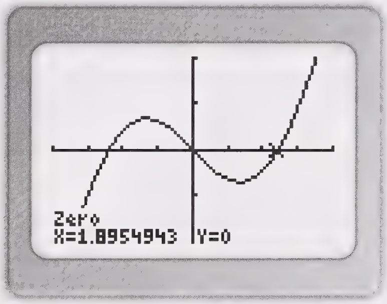
b. The graph of $y = x - 2 \sin x$ (shown in question 4.a.) has three x-intercepts that correspond to the solutions of the equation $2 \sin x = x$. Use the Zero feature to find the x-intercepts.



```
Zero
X=0            Y=0
```



```
Zero
X=⁻1.895494  Y=0
```



```
Zero
X=1.8954943  Y=0
```

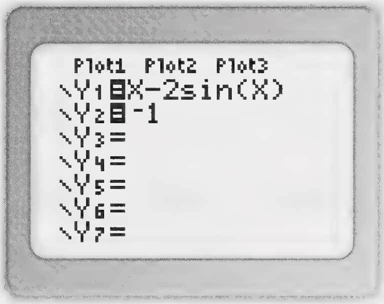
The solutions to the equation $2 \sin x = x$ are 0 and ± 1.895 rad.

c. $1+x-2\sin x=0$

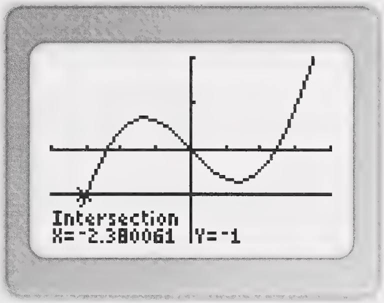
$x-2\sin x=-1$

You can use the graph of $y=x-2\sin x$ to see how many roots the equation $1+x-2\sin x=0$ has by counting the number of times the graph of $y=x-2\sin x$ intersects with the line $y=-1$.

d. To find the solutions to $1+x-2\sin x=0$, first graph the functions $y=x-2\sin x$ and $y=-1$ on your graphing calculator. Then use the Intersect feature to determine where the two graphs intersect.



```
Plot1 Plot2 Plot3
\Y1=X-2sin(X)
\Y2=-1
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



```
Intersection
X=-2.380061  Y=-1
```

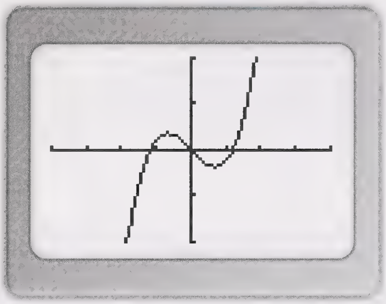
There is only one intersection; thus, the root of the equation $1+x-2\sin x=0$ is approximately -2.380.

5. One exact root of $x^3-\sin x=0$ is 0. The other roots are approximately ± 0.929.

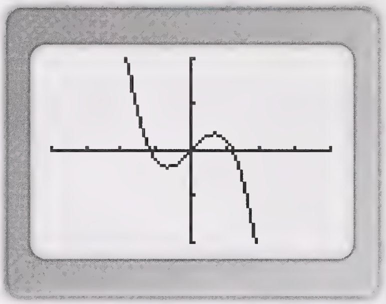
6. The two functions are $y=x^3$ and $y=\sin x$.

Section 2: Activity 3 (continued)

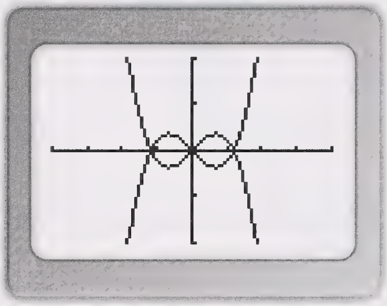
7. The equation $\sin x - x^3 = 0$ would have the same roots as $x^3 - \sin x = 0$. You can transform one equation into the other by multiplying both sides of one equation by -1. This will change the associated graphs by reflecting them in the x-axis, but it will not change the x-intercepts. (Refer to the following displays.)



$y = x^3 - \sin x$



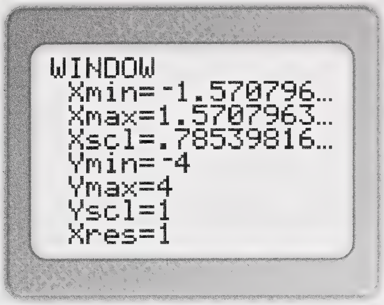
$y = \sin x - x^3$



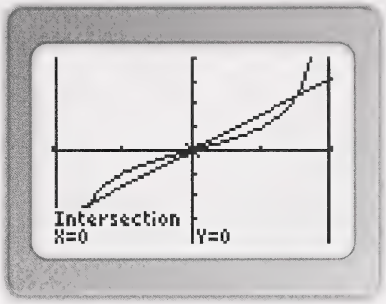
Both Graphs

3. Textbook questions 10 and 13 of "Applications and Problem Solving," p. 257

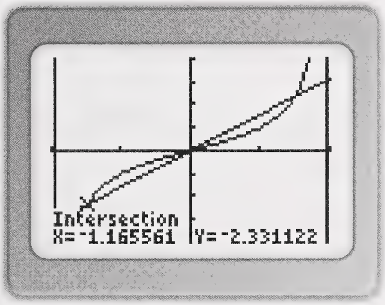
10. **a.** Graph $y = \tan x$ and $y = 2x$; then find the points of intersection.



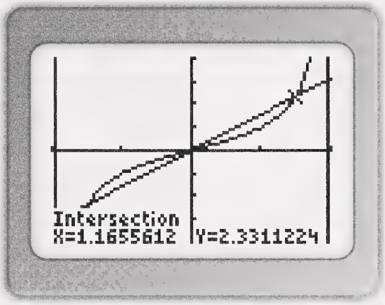
```
WINDOW
 Xmin=-1.570796…
 Xmax=1.5707963…
 Xscl=.78539816…
 Ymin=-4
 Ymax=4
 Yscl=1
 Xres=1
```



```
Intersection
X=0          Y=0
```



```
Intersection
X=-1.165561  Y=-2.331122
```



```
Intersection
X=1.1655612  Y=2.3311224
```

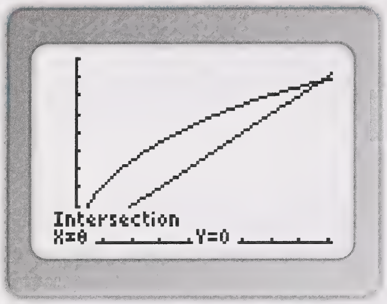
The solutions to $\tan x = 2x$ in the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ are 0 and ± 1.1656, correct to four decimal places.

b. The graph of $\tan x = 2x$ in the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ has three solutions. These are not the only solutions to $\tan x = 2x$. There will be one solution for each period of $\tan x$. Because the range for each period of tangent is the set of real numbers, the line $y = 2x$ will assume a value in the range of $\tan x$ for each period, thus giving a solution to $\tan x = 2x$ in each period. To check this graphically, change the graphing window to several different values and note where other solutions occur. There is an infinite number of solutions.

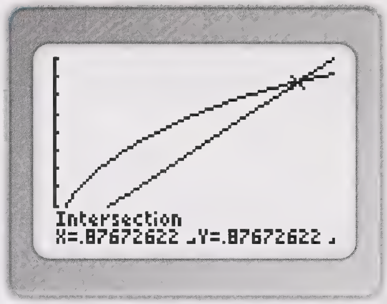
13. **a.** Sine never exceeds 1; so, the domain used to solve the equation $\sqrt{\sin x} = x$ cannot exceed 1. Also, $\sqrt{\sin x}$ is the principal (or non-negative) root of sin x; so, since $\sqrt{\sin x} = x$, x cannot be negative. Therefore, a suitable domain is $0 \le x \le 1$. The graph of $\sqrt{\sin x}$ is always non-negative, and neither $\sqrt{\sin x}$ nor x will exceed 1; so, a suitable range is $0 \le y \le 1$.

Section 2: Activity 3 (continued)

b.



Intersection
X=0 Y=0



Intersection
X=.87672622 Y=.87672622

The solutions to the equation $\sqrt{\sin x} = x$ are 0 and 0.8767, correct to four decimal places.

c.

LS	RS
$\sqrt{\sin x}$	x
$=\sqrt{\sin 0}$	$=0$
$=\sqrt{0}$	
$=0$	

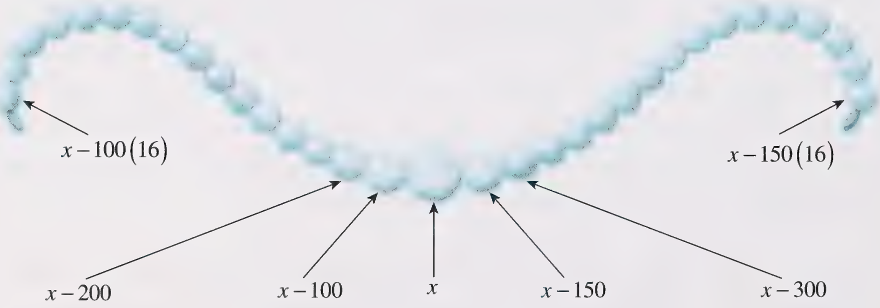
LS = RS

LS	RS
$\sqrt{\sin x}$	x
$\doteq\sqrt{\sin 0.8767}$	$\doteq 0.8767$
$\doteq 0.7686$	
$\doteq 0.8767$	

LS $\doteq$ RS

4. Textbook question "LOGIC POWER," p. 257

Let x be the value of the middle pearl. Because there are 33 pearls altogether, there are 16 pearls to the left of the middle pearl and 16 pearls to the right of the middle pearl.



The values of the 16 pearls on the left are $x-100, x-200, \ldots, x-100(16)$.

The values of the 16 pearls on the right are $x-150, x-300, \ldots, x-150(16)$.

The total value of all 33 pearls is \$65 000.

$$\begin{array}{l} \therefore\ 65\,000 = x + (x-100) + (x-200) + \ldots + (x-1600) \\ \qquad\qquad\quad + (x-150) + (x-300) + \ldots + (x-2400) \\ \hline 65\,000 = x + (2x-250) + (2x-500) + \ldots + (2x-4000) \end{array}$$

$$65\,000 = x + 16(2x) - (250 + 500 + \ldots + 4000)$$

$$65\,000 = x + 32x - \left[\frac{n}{2}(a + t_n)\right] \longleftarrow \text{sum of a series, where } n = 16,\ a = 250,\ \text{and } t_n = 4000$$

$$65\,000 = 33x - \left[\frac{16}{2}(250 + 4000)\right]$$

$$65\,000 = 33x - 34\,000$$

$$99\,000 = 33x$$

$$33x = 99\,000$$

$$x = 3000$$

The value of the middle pearl is \$3000.

Section 2: Activity 4

1. a. Textbook questions a. to c. of "Explore: Use Inductive Reasoning," p. 258

a.

Measure of ∠A	**0°**	**30°**	**45°**	**60°**	**90°**
sin A	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\sin^2 A + \cos^2 A$	1	1	1	1	1

b. The value of $\sin^2 A + \cos^2 A$ will always be 1.

Section 2: Activity 4 (continued)

c. Answers may vary. A sample answer is given.

Measure of $\angle A$	$\frac{4\pi}{3}$	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	**3.2**	**-2.4**
$\sin A$	0.8660	-0.5	-1	-0.0584	-0.6755
$\cos A$	-0.5	0.8660	0	-0.9983	-0.7374
$\sin^2 A + \cos^2 A$	1	1	1	1	1

No counterexamples can be found.

b. Textbook questions 1 to 5 of "Inquire," p. 258

1. The same result will hold in all four quadrants. Because sine and cosine are defined in terms of a right triangle, the Pythagorean relationship among the sides will hold. Sine and cosine are simply expressions of the lengths of the sides of a right triangle, where the hypotenuse has a length of 1.

2. Every point on the unit circle will satisfy the equation $x^2 + y^2 = 1$. The *y*-coordinate of the point on the unit circle, where the angle's terminal arm crosses the circle, is the sine for that angle. Similarly, cosine is the *x*-coordinate. Because $(\cos A, \sin A)$ represents a point on the unit circle, substituting for *x* and *y* in the equation yields $(\cos A)^2 + (\sin A)^2 = 1$ or $\cos^2 A + \sin^2 A = 1$ or $\sin^2 A + \cos^2 A = 1$.

3. The proof holds in all quadrants because the equation of the unit circle and the definitions of sine and cosine do not depend on the quadrant in which the terminal arm of the angle lies.

4. The two forms are equivalent for $\frac{\pi}{6}$ as the following calculations show.

LS	RS
$\dfrac{g \sec \theta}{\omega^2}$	$\dfrac{g \tan \theta}{\omega^2 \sin \theta}$
$= \dfrac{g \sec \frac{\pi}{6}}{\omega^2}$	$= \dfrac{g \tan \frac{\pi}{6}}{\omega^2 \sin \frac{\pi}{6}}$
$= \dfrac{g\left(\frac{2\sqrt{3}}{3}\right)}{\omega^2}$	$= \dfrac{g\left(\frac{\sqrt{3}}{3}\right)}{\omega^2 \left(\frac{1}{2}\right)}$
$= \dfrac{2\sqrt{3}g}{3\omega^2}$	$= \dfrac{2\sqrt{3}g}{3\omega^2}$

LS = RS

5. Prove that $\frac{g \tan \theta}{\omega^2 \sin \theta} = \frac{g \sec \theta}{\omega^2}$.

$$\textbf{LS} = \frac{g \tan \theta}{\omega^2 \sin \theta}$$

$$= \frac{g\left(\frac{\sin \theta}{\cos \theta}\right)}{\omega^2 \sin \theta} \quad \longleftarrow \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{g\left(\frac{\cancel{\sin \theta}}{\cos \theta \cancel{\sin \theta}}\right)}{\omega^2}$$

$$= \frac{g\left(\frac{1}{\cos \theta}\right)}{\omega^2}$$

$$= \frac{g \sec \theta}{\omega^2} \quad \longleftarrow \sec\theta = \frac{1}{\cos\theta}$$

$$= \textbf{RS}$$

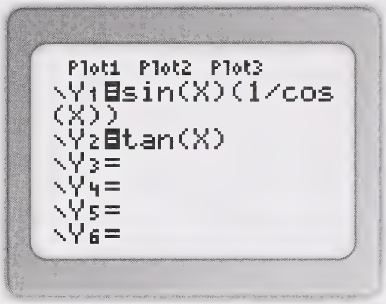
The expression $\frac{g \tan \theta}{\omega^2 \sin \theta}$ is defined for the set of real numbers except when $\sin \theta = 0$ or $\cos \theta = 0$; that is when $\theta = k\pi$ and $\theta = \frac{\pi}{2} + k\pi$, where k is any integer. These can be combined as $\theta = \frac{k\pi}{2}$, where k is any integer.

The expression $\frac{g \sec \theta}{\omega^2}$ is defined for the set of real numbers except when $\cos \theta = 0$; that is when $\theta = \frac{\pi}{2} + k\pi$, where k is any integer.

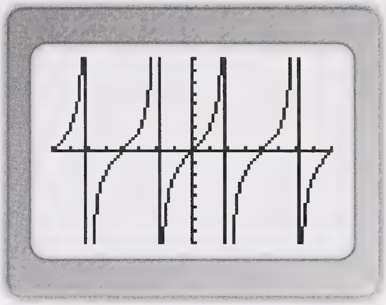
Section 2: Activity 4 (continued)

2. Textbook questions 1, 4, and 5 of "Practice," p. 264

1. Graph $y = \sin\theta \sec\theta$ as Y_1 and $y = \tan\theta$ as Y_2.



```
 Plot1 Plot2 Plot3
\Y1=sin(X)(1/cos
(X))
\Y2=tan(X)
\Y3=
\Y4=
\Y5=
\Y6=
```

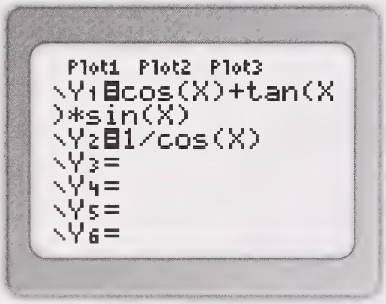


Because identical graphs are obtained for both functions, $\sin\theta \sec\theta = \tan\theta$ is likely an identity.

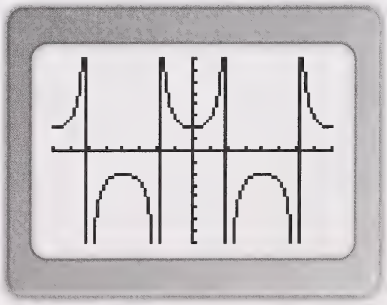
$$\begin{aligned}\textbf{LS} &= \sin\theta \sec\theta \\ &= \sin\theta \frac{1}{\cos\theta} \\ &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta \\ &= \textbf{RS}\end{aligned}$$

Both $\sin\theta \sec\theta$ and $\tan\theta$ are undefined when $\cos\theta = 0$. Thus, $\sin\theta \sec\theta = \tan\theta$ is an identity for any real number θ such that $\cos\theta \neq 0$.

4. Graph $y = \cos A + \tan A \sin A$ as Y_1 and $y = \sec A$ as Y_2.



```
 Plot1 Plot2 Plot3
\Y1=cos(X)+tan(X
)*sin(X)
\Y2=1/cos(X)
\Y3=
\Y4=
\Y5=
\Y6=
```

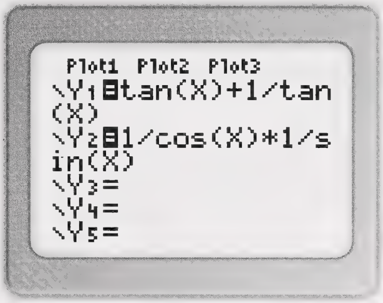


Because identical graphs are obtained for both functions, $\cos A + \tan A \sin A = \sec A$ is likely an identity.

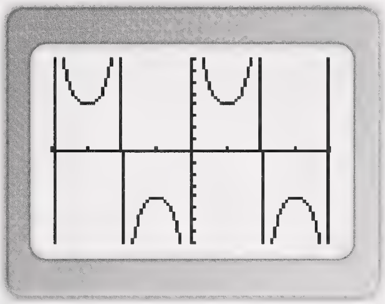
$$\begin{aligned}
\mathbf{LS} &= \cos A + \tan A \sin A \\
&= \cos A + \frac{\sin A}{\cos A}\sin A \qquad \leftarrow \frac{\tan A}{\cos A} = \sin A \\
&= \cos A + \frac{\sin^2 A}{\cos A} \\
&= \frac{\cos^2 A + \sin^2 A}{\cos A} \\
&= \frac{1}{\cos A} \qquad \leftarrow \cos^2 A + \sin^2 A \\
&= \sec A \\
&= \mathbf{RS}
\end{aligned}$$

Both $\cos A + \tan A \sin A$ and $\sec A$ are undefined when $\cos A = 0$. Thus, $\cos A + \tan A \sin A = \sec A$ is an identity for any real number A such that $\cos A \neq 0$.

5. Graph $y = \tan\theta + \cot\theta$ as Y_1 and $y = \sec\theta\csc\theta$ as Y_2.



```
Plot1 Plot2 Plot3
\Y1■tan(X)+1/tan
(X)
\Y2■1/cos(X)*1/s
in(X)
\Y3=
\Y4=
\Y5=
```



Because identical graphs are obtained for both functions, $\tan\theta + \cot\theta = \sec\theta\csc\theta$ is likely an identity.

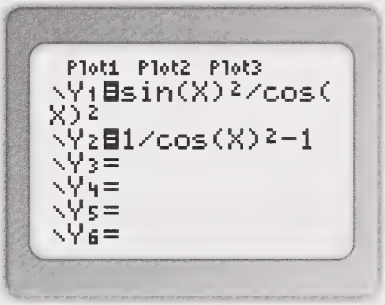
Section 2: Activity 4 (continued)

$$
\begin{aligned}
\textbf{LS} &= \tan\theta + \cot\theta \\
&= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} && \longleftarrow \tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta = \frac{\cos\theta}{\sin\theta} \\
&= \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta} \\
&= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \\
&= \frac{1}{\sin\theta\cos\theta} && \longleftarrow \cos^2\theta + \sin^2\theta = 1 \\
&= \frac{1}{\cos\theta} \times \frac{1}{\sin\theta} \\
&= \sec\theta\csc\theta \\
&= \textbf{RS}
\end{aligned}
$$

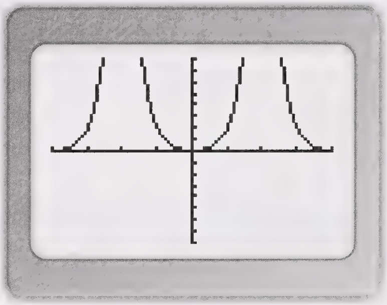
Both $\tan\theta + \cot\theta$ and $\sec\theta\csc\theta$ are undefined when $\cos\theta = 0$ or when $\sin\theta = 0$. Thus, $\tan\theta + \cot\theta = \sec\theta\csc\theta$ is an identity for any real number θ such that $\cos\theta \neq 0$ and $\sin\theta \neq 0$.

3. Textbook questions 6, 8, and 10 of "Practice," p. 264

6.



```
Plot1 Plot2 Plot3
\Y1■sin(X)²/cos(
X)²
\Y2■1/cos(X)²-1
\Y3=
\Y4=
\Y5=
\Y6=
```

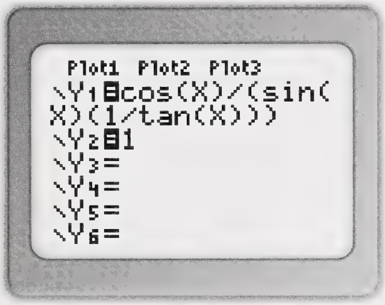


Because identical graphs are obtained for both functions, $\frac{\sin^2\theta}{\cos^2\theta} = \sec^2\theta - 1$ is likely an identity.

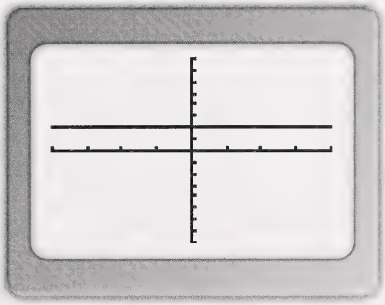
$$\begin{aligned}
\textbf{RS} &= \sec^2\theta - 1\\
&= \left(1+\tan^2\theta\right) - 1 \qquad \longleftarrow 1+\tan^2\theta = \sec^2\theta\\
&= \tan^2\theta\\
&= \left(\frac{\sin\theta}{\cos\theta}\right)^2 \qquad \longleftarrow \tan\theta = \frac{\sin\theta}{\cos\theta}\\
&= \frac{\sin^2\theta}{\cos^2\theta}\\
&= \textbf{LS}
\end{aligned}$$

Both $\frac{\sin^2\theta}{\cos^2\theta}$ and $\sec^2\theta - 1$ are undefined when $\cos\theta = 0$. Thus, $\frac{\sin^2\theta}{\cos^2\theta} = \sec^2\theta - 1$ is an identity for any real number θ such that $\cos\theta \neq 0$.

8.



```
 Plot1 Plot2 Plot3
\Y1■cos(X)/(sin(
X)(1/tan(X)))
\Y2■1
\Y3=
\Y4=
\Y5=
\Y6=
```



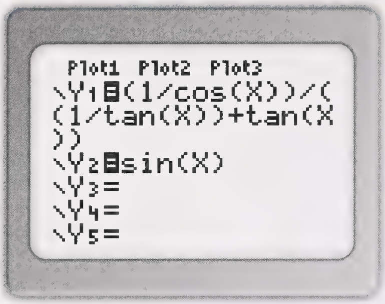
Because identical graphs are obtained for both functions, $\frac{\cos x}{\sin x \cot x} = 1$ is likely an identity.

$$\begin{aligned}
\textbf{LS} &= \frac{\cos x}{\sin x \cot x}\\
&= \frac{\cos x}{\cancel{\sin x}\left(\frac{\cos x}{\cancel{\sin x}}\right)} \qquad \longleftarrow \cot x = \frac{\cos x}{\sin x}\\
&= \frac{\cos x}{\cos x}\\
&= 1\\
&= \textbf{RS}
\end{aligned}$$

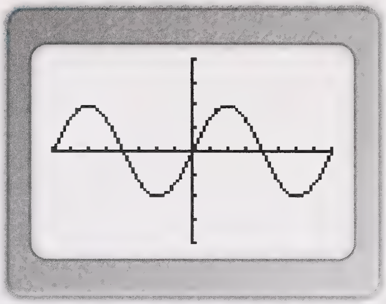
The expression $\frac{\cos x}{\sin x \cot x}$ is undefined when $\sin x = 0$ and $\cot x = 0$. Thus, $\frac{\cos x}{\sin x \cot x} = 1$ is an identity for any real number x such that $\cot x \neq 0$ and $\sin x \neq 0$.

Section 2: Activity 4 (continued)

10.



```
Plot1 Plot2 Plot3
\Y1■(1/cos(X))/(
(1/tan(X))+tan(X
))
\Y2■sin(X)
\Y3=
\Y4=
\Y5=
```



Because identical graphs are obtained for both functions, $\frac{\sec x}{\cot x+\tan x}=\sin x$ is likely an identity.

$$\mathbf{LS}=\frac{\sec x}{\cot x+\tan x}$$

$$=\frac{\sec x}{\frac{\cos x}{\sin x}+\frac{\sin x}{\cos x}} \quad \longleftarrow \cot x=\frac{\cos x}{\sin x} \text{ and } \tan x=\frac{\sin x}{\cos x}$$

$$=\frac{\sec x}{\frac{\cos^2 x}{\sin x\cos x}+\frac{\sin^2 x}{\sin x\cos x}}$$

$$=\frac{\sec x}{\frac{\cos^2 x+\sin^2 x}{\sin x\cos x}}$$

$$=\sec x\times\frac{\sin x\cos x}{\cos^2 x+\sin^2 x}$$

$$=\frac{1}{\cos x}\times\frac{\sin x\cos x}{1} \quad \longleftarrow \text{[illegible]}$$

$$=\frac{\sin x\,\cancel{\cos x}}{\cancel{\cos x}}$$

$$=\sin x$$

$$=\mathbf{RS}$$

The expression $\frac{\sec x}{\cot x+\tan x}$ is undefined when $\cos x=0$ and $\sin x=0$. Thus, $\frac{\sec x}{\cot x+\tan x}=\sin x$ is an identity for any real number x such that $\cos x\neq 0$ and $\sin x\neq 0$.

4. Textbook questions 16, 17, and 19 of "Practice," p. 264

16. a.

LS	RS
$\frac{1+\cos\theta}{\sin\theta}$	$\frac{\sin\theta}{1-\cos\theta}$
$=\frac{1+\cos 30^\circ}{\sin 30^\circ}$	$=\frac{\sin 30^\circ}{1-\cos 30^\circ}$
$=\frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}}$	$=\frac{\frac{1}{2}}{1-\frac{\sqrt{3}}{2}}$
$=\left(1+\frac{\sqrt{3}}{2}\right)\times\frac{2}{1}$	$=\frac{\frac{1}{2}}{1-\frac{\sqrt{3}}{2}}\times\frac{1+\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}$
$=2+\frac{\cancel{2}\sqrt{3}}{\cancel{2}}$	$=\frac{\frac{1}{2}+\frac{\sqrt{3}}{4}}{1-\frac{3}{4}}$
$=2+\sqrt{3}$	$=\frac{\frac{2+\sqrt{3}}{\cancel{4}}}{\frac{1}{\cancel{4}}}$
	$=2+\sqrt{3}$

LS = RS

b.

$$\begin{aligned}
\textbf{LS} &= \frac{1+\cos\theta}{\sin\theta}\\
&= \frac{1+\cos\theta}{\sin\theta}\times\frac{1-\cos\theta}{1-\cos\theta}\\
&= \frac{1-\cos^2\theta}{\sin\theta(1-\cos\theta)}\\
&= \frac{\sin^2\theta}{\sin\theta(1-\cos\theta)}\\
&= \frac{\sin\theta}{1-\cos\theta}\\
&= \textbf{RS}
\end{aligned}$$

c. The left side of the equation has the restriction $\sin\theta\neq 0$. So, $\theta\neq k\pi$, where k is any integer.

The right side of the equation has the restriction $1-\cos\theta\neq 0$ or $\cos\theta\neq 1$. So, $\theta\neq 2k\pi$, where k is any integer.

Therefore, $\frac{1+\cos\theta}{\sin\theta}=\frac{\sin\theta}{1-\cos\theta}$ is an identity with restriction $\theta\neq k\pi$, where k is any integer.

Section 2: Activity 4 (continued)

17. **a.**

LS	RS
$\frac{1+\tan\theta}{1+\cot\theta}$	$\tan\theta$
$=\frac{1+\tan 30^\circ}{1+\cot 30^\circ}$	$=\tan 30^\circ$
$=\frac{1+\frac{\sqrt{3}}{3}}{1+\sqrt{3}}$	$=\frac{\sqrt{3}}{3}$
$=\frac{1+\frac{\sqrt{3}}{3}}{1+\sqrt{3}}\times\frac{1-\sqrt{3}}{1-\sqrt{3}}$	
$=\frac{1-\sqrt{3}+\frac{\sqrt{3}}{3}-\frac{3}{3}}{1-3}$	
$=\frac{-\frac{3\sqrt{3}}{3}+\frac{\sqrt{3}}{3}}{-2}$	
$=\frac{-\frac{\cancel{2}\sqrt{3}}{3}}{-\cancel{2}}$	
$=\frac{\sqrt{3}}{3}$	

LS = RS

b.

$$
\begin{aligned}
\textbf{LS} &= \frac{1+\tan\theta}{1+\cot\theta}\\
&= \frac{1+\frac{\sin\theta}{\cos\theta}}{1+\frac{\cos\theta}{\sin\theta}}\\
&= \frac{\frac{\cos\theta+\sin\theta}{\cos\theta}}{\frac{\sin\theta+\cos\theta}{\sin\theta}}\\
&= \frac{\cancel{\cos\theta+\sin\theta}}{\cos\theta}\times\frac{\sin\theta}{\cancel{\sin\theta+\cos\theta}}\\
&= \frac{\sin\theta}{\cos\theta}\\
&= \tan\theta\\
&= \textbf{RS}
\end{aligned}
$$

c. The left side of the equation has the restrictions $1+\cot\theta\neq 0$ or $\cot\theta\neq -1$ and $\sin\theta\neq 0$. So, $\theta\neq\frac{3\pi}{4}+k\pi$ and $\theta\neq k\pi$, where k is any integer.

The right side of the equation has the restriction $\cos\theta\neq 0$. So, $\theta\neq\frac{\pi}{2}+k\pi$, where k is any integer. This restriction can be combined with $\theta\neq k\pi$ as $\theta\neq\frac{k\pi}{2}$, where k is any integer.

Therefore, $\frac{1+\tan\theta}{1+\cot\theta}=\tan\theta$ is an identity with restrictions $\theta\neq\frac{k\pi}{2}$ and $\theta\neq\frac{3\pi}{4}+k\pi$, where k is any integer.

19. **a.**

LS	RS
$\frac{\sec\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}$	$\cot\theta$
$=\frac{\sec 30°}{\sin 30°}-\frac{\sin 30°}{\cos 30°}$	$=\cot 30°$
$=\frac{\frac{2\sqrt{3}}{3}}{\frac{1}{2}}-\frac{\frac{1}{\cancel{2}}}{\frac{\sqrt{3}}{\cancel{2}}}$	$=\sqrt{3}$
$=\frac{4\sqrt{3}}{3}-\frac{1}{\sqrt{3}}$	
$=\frac{4\sqrt{3}}{3}-\frac{\sqrt{3}}{3}$	
$=\frac{3\sqrt{3}}{3}$	
$=\sqrt{3}$	

LS = RS

b. $$\textbf{LS}=\frac{\sec\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}$$

$$=\frac{\frac{1}{\cos\theta}}{\sin\theta}-\frac{\sin\theta}{\cos\theta}$$

$$=\frac{1}{\sin\theta\cos\theta}-\frac{\sin\theta}{\cos\theta}$$

$$=\frac{1}{\sin\theta\cos\theta}-\frac{\sin^2\theta}{\sin\theta\cos\theta}$$

$$=\frac{1-\sin^2\theta}{\sin\theta\cos\theta}$$

$$=\frac{\cos^2\theta}{\sin\theta\cos\theta}$$

$$=\frac{\cos\theta}{\sin\theta}$$

$$=\cot\theta$$

$$=\textbf{RS}$$

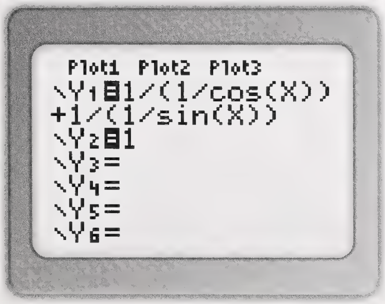
c. The left side of the equation has the restrictions $\sin\theta\neq 0$ and $\cos\theta\neq 0$. So, $\theta\neq k\pi$ and $\theta\neq\frac{k\pi}{2}$, where k is any integer.

The right side of the equation has the restriction $\sin\theta\neq 0$. So, $\theta\neq k\pi$, where k is any integer.

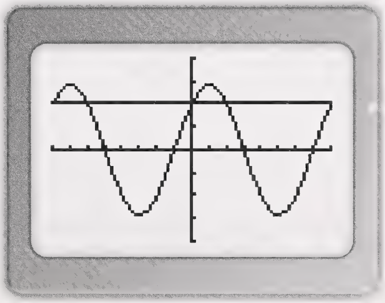
Therefore, $\frac{\sec\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}=\cot\theta$ is an identity with restriction $\theta\neq\frac{k\pi}{2}$, where k is any integer.

5. Textbook questions 22, 23, 24, 27, 29, and 32 of "Practice," p. 264

22. Graph $y=\frac{1}{\sec x}+\frac{1}{\csc x}$ as Y_1 and $y=1$ as Y_2.



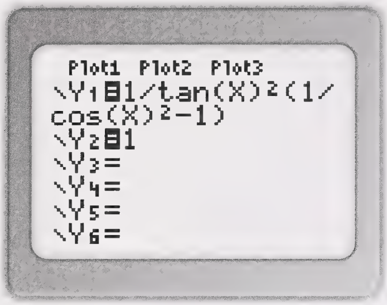
```
Plot1 Plot2 Plot3
\Y1=1/(1/cos(X))
+1/(1/sin(X))
\Y2=1
\Y3=
\Y4=
\Y5=
\Y6=
```



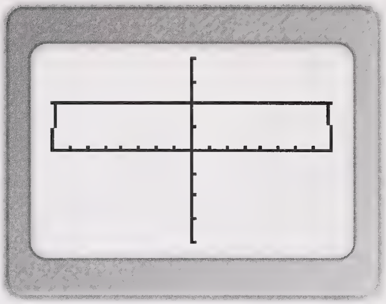
Because the two graphs do not coincide, the equation $\frac{1}{\sec x}+\frac{1}{\csc x}=1$ is not an identity.

Section 2: Activity 4 (continued)

23. Graph $y = \cot^2 x\left(\sec^2 x - 1\right)$ as Y_1 and $y = 1$ as Y_2.



```
Plot1 Plot2 Plot3
\Y1=1/tan(X)²(1/
cos(X)²-1)
\Y2=1
\Y3=
\Y4=
\Y5=
\Y6=
```

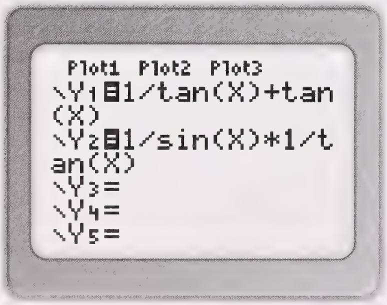


The equation appears to be an identity.

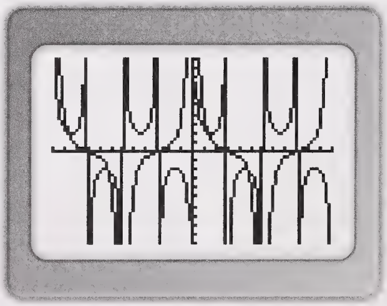
$$\begin{aligned}
\textbf{LS} &= \cot^2 x\left(\sec^2 x - 1\right)\\
&= \cot^2 x\left(\tan^2 x\right)\\
&= \frac{1}{\tan^2 x}\left(\tan^2 x\right)\\
&= 1\\
&= \textbf{RS}
\end{aligned}$$

Because the left side is not defined when $\sin x = 0$ or $\cos x = 0$, the domain of the identity will be all real numbers x except $x = k\pi$ and $x = \frac{\pi}{2} + k\pi$, where k is any integer. These restrictions can be combined. Therefore, the domain is all real numbers except $x = \frac{k\pi}{2}$, where k is any integer.

24. Graph $y = \cot x + \tan x$ as Y_1 and $y = \csc x \cot x$ as Y_2.



```
Plot1 Plot2 Plot3
\Y1=1/tan(X)+tan
(X)
\Y2=1/sin(X)*1/t
an(X)
\Y3=
\Y4=
\Y5=
```



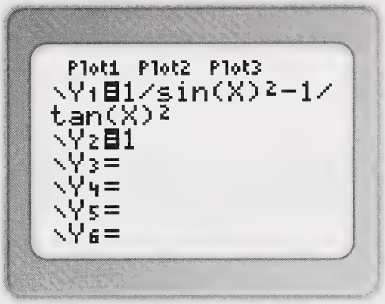
Because the graphs are not identical, the equation $\cot x + \tan x = \csc x \cot x$ is not an identity.

27. $\csc^2 x - \cot^2 x = \left(\cot^2 x + 1\right) - \cot^2 x$

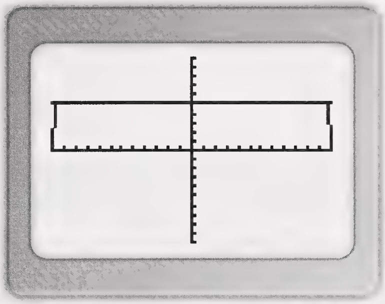
$$= 1$$

Check

Graph $y = \csc^2 x - \cot^2 x$ as Y_1 and $y = 1$ as Y_2.



```
 Plot1  Plot2  Plot3
\Y1■1/sin(X)²-1/
tan(X)²
\Y2■1
\Y3=
\Y4=
\Y5=
\Y6=
```

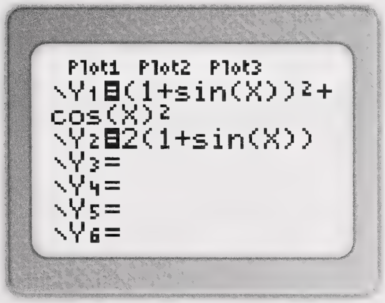


29. $(1+\sin x)^2 + \cos^2 x = \left(1 + 2\sin x + \sin^2 x\right) + \cos^2 x$

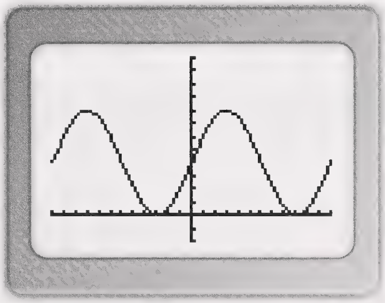
$$= 1 + 2\sin x + \sin^2 x + \cos^2 x$$
$$= 1 + 2\sin x + 1$$
$$= 2 + 2\sin x$$
$$= 2(1+\sin x)$$

Check

Graph $y = (1+\sin x)^2 + \cos^2 x$ as Y_1 and $y = 2(1+\sin x)$ as Y_2.



```
 Plot1  Plot2  Plot3
\Y1■(1+sin(X))²+
cos(X)²
\Y2■2(1+sin(X))
\Y3=
\Y4=
\Y5=
\Y6=
```

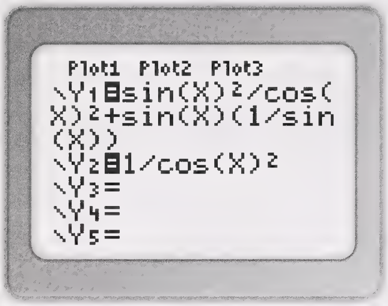


Section 2: Activity 4 (continued)

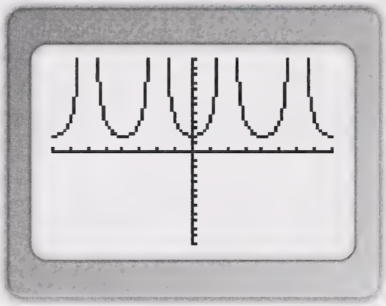
32. $$\frac{\sin^2 x}{\cos^2 x} + \sin x \csc x = \frac{\sin^2 x}{\cos^2 x} + \cancel{\sin x}\left(\frac{1}{\cancel{\sin x}}\right)$$
$$= \tan^2 x + 1$$
$$= \sec^2 x$$

Check

Graph $y = \frac{\sin^2 x}{\cos^2 x} + \sin x \csc x$ as Y_1 and $y = \sec^2 x$ as Y_2.

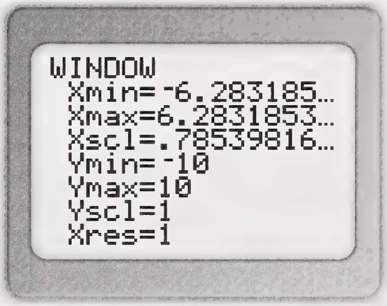


```
 Plot1  Plot2  Plot3
\Y1=sin(X)²/cos(
X)²+sin(X)(1/sin
(X))
\Y2=1/cos(X)²
\Y3=
\Y4=
\Y5=
```

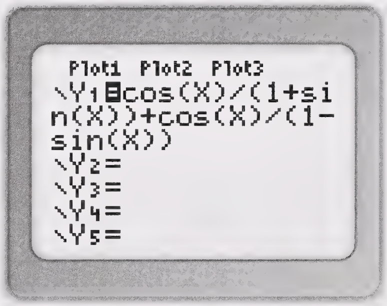


6. Textbook questions 36, 39, and 42 of “Applications and Problem Solving,” p. 265

36. a.

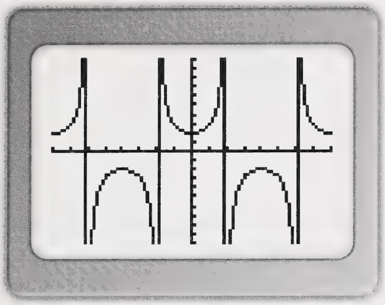


```
WINDOW
 Xmin=-6.283185…
 Xmax=6.2831853…
 Xscl=.78539816…
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
```



```
 Plot1  Plot2  Plot3
\Y1=cos(X)/(1+si
n(X))+cos(X)/(1-
sin(X))
\Y2=
\Y3=
\Y4=
\Y5=
```

b.



The graph suggests that $f(x) = 2\sec x$.

c.

LS	RS
$\dfrac{\cos x}{1+\sin x} + \dfrac{\cos x}{1-\sin x}$	$2\sec x$
$= \dfrac{\cos x}{1+\sin x} \times \dfrac{1-\sin x}{1-\sin x} + \dfrac{\cos x}{1-\sin x} \times \dfrac{1+\sin x}{1+\sin x}$	
$= \dfrac{\cos x - \cos x \sin x}{1-\sin^2 x} + \dfrac{\cos x + \cos x \sin x}{1-\sin^2 x}$	
$= \dfrac{\cos x - \cos x \sin x + \cos x + \cos x \sin x}{1-\sin^2 x}$	
$= \dfrac{2\cos x}{1-\sin^2 x}$	
$= \dfrac{2\cos x}{\cos^2 x} \quad \leftarrow 1-\sin^2 x = \cos^2 x$	
$= \dfrac{2}{\cos x}$	
$= 2 \times \dfrac{1}{\cos x}$	
$= 2\sec x$	

LS = RS

d. The domain of $\frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x}$ is all real numbers except those making $\sin x = \pm 1$. So, $x \neq \frac{\pi}{2} + k\pi$, where k is any integer.

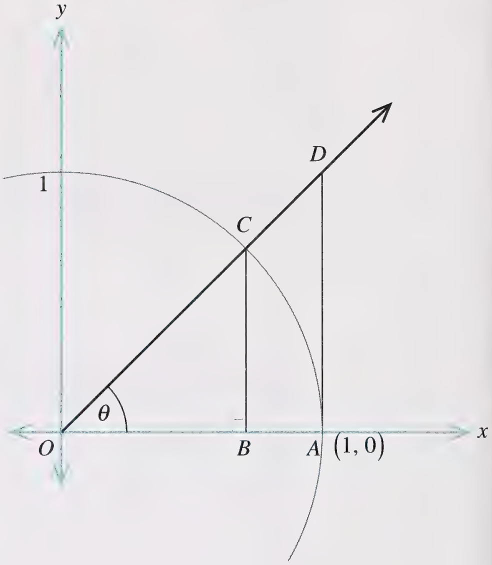
The domain of $2\sec x$ is the set of real numbers except those making $\cos x = 0$. So, $x \neq \frac{\pi}{2} + k\pi$, where k is any integer.

Section 2: Activity 4 (continued)

39. Prove $\tan^2\theta+1=\sec^2\theta$.

LS	RS
$\tan^2\theta+1$	$\sec^2\theta$
$=\dfrac{(BC)^2}{(BO)^2}+1$	$=\dfrac{(CO)^2}{(BO)^2}$
$=\dfrac{(BC)^2+(BO)^2}{(BO)^2}$	
$=\dfrac{(CO)^2}{(BO)^2}$	

LS = RS



Prove $1+\cot^2\theta=\csc^2\theta$.

LS	RS
$1+\cot^2\theta$	$\csc^2\theta$
$=1+\dfrac{(BO)^2}{(BC)^2}$	$=\dfrac{(CO)^2}{(BC)^2}$
$=\dfrac{(BO)^2+(BC)^2}{(BC)^2}$	
$=\dfrac{(CO)^2}{(BC)^2}$	

LS = RS

The name Pythagorean identities is appropriate because they involve using the Pythagorean Theorem in their proofs.

42.
$$\begin{aligned}\sqrt{\left(4-x^2\right)^3} &= \sqrt{\left[4-(2\cos t)^2\right]^3}\\ &= \sqrt{\left(4-4\cos^2 t\right)^3}\\ &= \sqrt{\left[4\left(1-\cos^2 t\right)\right]^3}\\ &= \sqrt{\left(4\sin^2 t\right)^3}\\ &= \sqrt{64\sin^6 t}\\ &= \sqrt{64}\times\sqrt{\sin^6 t}\\ &= 8\sin^3 t\end{aligned}$$

The restriction $0 \le t \le \pi$ is needed to keep a one-to-one correspondence between values of x and t. You want to be able to relate a value of x to a unique value of t.

7. Textbook question "WORD POWER," p. 265

Answers may vary. A sample answer is given.

D	A	F	T
A	B	L	E
F	L	E	A
T	E	A	K

If you want to see how technology could help you answer this question, open the file titled *dbek.xls* from the Module_Work folder on the Pure Mathematics 30 Companion CD. Press the "Start Search" button, and watch the possible solutions be tested. The unique solution to the problem will be displayed within 10 s or 15 s of searching. If you are interested in how the results are found, use the Visual Basic Editor in *Excel*® to examine the coding.

Section 2: Activity 5

1. a. Textbook questions 1 and 2 of Investigation 1, “Investigating $\cos(\mathrm{A}-\mathrm{B})$,” p. 266

1.

LS	RS
$\cos(A-B)$	$\cos A-\cos B$
$=\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$	$=\cos\frac{\pi}{3}-\cos\frac{\pi}{6}$
$=\cos\frac{\pi}{6}$	$=\frac{1}{2}-\frac{\sqrt{3}}{2}$
$=\frac{\sqrt{3}}{2}$	$=\frac{1-\sqrt{3}}{2}$

LS $\neq$ RS

Because the equation is false for these two values, it cannot be an identity.

2. a. On the unit circle, the coordinates of a point on the terminal arm of an angle, θ, are always $(\cos\theta, \sin\theta)$. This is because $\cos\theta=\frac{x}{r}$ and $\sin\theta=\frac{y}{r}$, where $r=1$. The coordinates for W are $(\cos A, \sin A)$, and the coordinates for R are $(\cos(A-B), \sin(A-B))$.

b. Chord XR is congruent to chord WZ because the angles intercepting these chords are both equal to $A-B$.

c.

$$d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

$$\begin{aligned}
XR&=\sqrt{[1-\cos(A-B)]^2+[0-\sin(A-B)]^2}\\
&=\sqrt{1-2\cos(A-B)+\cos^2(A-B)+\sin^2(A-B)}\\
&=\sqrt{1+[\sin^2(A-B)+\cos^2(A-B)]-2\cos(A-B)}\\
&=\sqrt{1+1-2\cos(A-B)}\\
&=\sqrt{2-2\cos(A-B)}
\end{aligned}$$

$$d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

$$\begin{aligned}
WZ&=\sqrt{(\cos A-\cos B)^2+(\sin A-\sin B)^2}\\
&=\sqrt{\left[\cos^2 A-2\cos A\cos B+\cos^2 B\right]+\left[\sin^2 A-2\sin A\sin B+\sin^2 B\right]}\\
&=\sqrt{\sin^2 A+\cos^2 A+\sin^2 B+\cos^2 B-\left(2\sin^2 A\sin B+2\cos A\cos B\right)}\\
&=\sqrt{1+1-(2\sin A\sin B+2\cos A\cos B)}\\
&=\sqrt{2-(2\sin A\sin B+2\cos A\cos B)}
\end{aligned}$$

$$\therefore\ XR=WZ$$

$$\sqrt{2-2\cos(A-B)}=\sqrt{2-(2\sin A\sin B+2\cos A\cos B)}$$

d.

$$\begin{aligned}
\sqrt{2-2\cos(A-B)}&=\sqrt{2-(2\sin A\sin B+2\cos A\cos B)}\\
\left[\sqrt{2-2\cos(A-B)}\right]^2&=\left[\sqrt{2-(2\sin A\sin B+2\cos A\cos B)}\right]^2\\
\cancel{2}-2\cos(A-B)&=\cancel{2}-(2\sin A\sin B+2\cos A\cos B)\\
\cancel{-2}\cos(A-B)&=\cancel{-2}(\sin A\sin B+\cos A\cos B)\\
\cos(A-B)&=\sin A\sin B+\cos A\cos B\\
\cos(A-B)&=\cos A\cos B+\sin A\sin B
\end{aligned}$$

e.

LS	RS
$\cos(A-B)$	$\cos A\cos B+\sin A\sin B$
$=\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$	$=\cos\frac{\pi}{3}\cos\frac{\pi}{6}+\sin\frac{\pi}{3}\sin\frac{\pi}{6}$
$=\cos\frac{\pi}{6}$	$=\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$
$=\frac{\sqrt{3}}{2}$	$=\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{4}$
	$=\frac{2\sqrt{3}}{4}$
	$=\frac{\sqrt{3}}{2}$

LS = RS

Section 2: Activity 5 (continued)

b. Textbook question 1 of Investigation 2, "Deriving an Identity for $\cos(A+B)$," p. 266

1. a.
$$\begin{aligned}\cos(A+B) &= \cos[A-(-B)]\\ &= \cos A\cos(-B)+\sin A\sin(-B)\\ &= \cos A\cos B+\sin A\sin(-B)\\ &= \cos A\cos B-\sin A\sin B\end{aligned}$$

$\cos(-B)=\cos B$

and

$\sin(-B)=-\sin B$

b.

LS	RS
$\cos(A+B)$	$\cos A\cos B-\sin A\sin B$
$=\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$	$=\cos\frac{\pi}{3}\cos\frac{\pi}{6}-\sin\frac{\pi}{3}\sin\frac{\pi}{6}$
$=\cos\frac{\pi}{2}$	$=\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$
$=0$	$=\frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4}$
	$=0$

LS = RS

c. Textbook questions 1 to 4 of Investigation 3, "Deriving Identities for $\sin(A+B)$, $\sin(A-B)$, $\tan(A+B)$, and $\tan(A-B)$," p. 267

1.
$$\begin{aligned}\sin(A+B) &= \cos[90°-(A+B)]\\ &= \cos[(90°-A)-B]\\ &= \cos(90°-A)\cos B+\sin(90°-A)\sin B\\ &= \sin A\cos B+\cos A\sin B\end{aligned}$$

$\sin\theta=\cos(90°-\theta)$

$\cos\theta=\sin(90°-\theta)$

2.
$$\begin{aligned}\sin(A-B) &= \sin[A+(-B)]\\ &= \sin A\cos(-B)+\cos A\sin(-B)\\ &= \sin A\cos B-\cos A\sin B\end{aligned}$$

3.

$$\begin{aligned}
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} && \begin{array}{l} \leftarrow \text{sum formula for [illegible]} \\ \leftarrow \text{sum formula for [illegible]} \end{array} \\
&= \frac{\frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B - \sin A \sin B}{\cos A \cos B}} && \leftarrow \text{Divide the numerato[illegible]tor by [illegible]} \\
&= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A \sin B}{\cos A \cos B}} \\
&= \frac{\tan A + \tan B}{1 - \frac{\sin A}{\cos A} \times \frac{\sin B}{\cos B}} \\
&= \frac{\tan A + \tan B}{1 - \tan A \tan B}
\end{aligned}$$

4.

$$\begin{aligned}
\tan(A-B) &= \tan[A+(-B)] \\
&= \frac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)} \\
&= \frac{\tan A - \tan B}{1 + \tan A \tan B} && \tan(-\theta) = -\tan\theta
\end{aligned}$$

d. Textbook question 1 of Investigation 4, "Double-Angle Identities," p. 267

1. a.

$$\begin{aligned}
\sin 2A &= \sin(A+A) \\
&= \sin A \cos A + \cos A \sin A \\
&= 2 \sin A \cos A
\end{aligned}$$

b.

$$\begin{aligned}
\cos 2A &= \cos(A+A) \\
&= \cos A \cos A - \sin A \sin A \\
&= \cos^2 A - \sin^2 A
\end{aligned}$$

c.

$$\begin{aligned}
\cos 2A &= \cos^2 A - \sin^2 A \\
&= \cos^2 A - (1 - \cos^2 A) \\
&= \cos^2 A - 1 + \cos^2 A \\
&= 2 \cos^2 A - 1
\end{aligned}$$

d.

$$\begin{aligned}
\cos 2A &= \cos^2 A - \sin^2 A \\
&= (1 - \sin^2 A) - \sin^2 A \\
&= 1 - 2 \sin^2 A
\end{aligned}$$

e.

$$\begin{aligned}
\tan 2A &= \tan(A+A) \\
&= \frac{\tan A + \tan A}{1 - \tan A \tan A} \\
&= \frac{2 \tan A}{1 - \tan^2 A}
\end{aligned}$$

Section 2: Activity 5 (continued)

2. Textbook questions 1, 3, and 6 of "Practice," p. 272

1. The expression $\sin 65^\circ \cos 35^\circ - \cos 65^\circ \sin 35^\circ$ has the same pattern as the right side of the identity $\sin(A-B) = \sin A \cos B - \cos A \sin B$, where $A = 65^\circ$ and $B = 35^\circ$.

$$\begin{aligned}\therefore\ \sin 65^\circ \cos 35^\circ - \cos 65^\circ \sin 35^\circ &= \sin(65^\circ - 35^\circ)\\ &= \sin 30^\circ\\ &= \frac{1}{2}\end{aligned}$$

3. The expression $\cos 25^\circ \cos 5^\circ - \sin 25^\circ \sin 5^\circ$ has the same pattern as the right side of the identity $\cos(A+B) = \cos A \cos B - \sin A \sin B$, where $A = 25^\circ$ and $B = 5^\circ$.

$$\begin{aligned}\therefore\ \cos 25^\circ \cos 5^\circ - \sin 25^\circ \sin 5^\circ &= \cos(25^\circ + 5^\circ)\\ &= \cos 30^\circ\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

6. The expression $\cos \frac{7\pi}{12} \cos \frac{\pi}{3} + \sin \frac{7\pi}{12} \sin \frac{\pi}{3}$ has the same pattern as the right side of the identity $\cos(A-B) = \cos A \cos B + \sin A \sin B$,where $A = \frac{7\pi}{12}$ and $B = \frac{\pi}{3}$.

$$\begin{aligned}\therefore\ \cos \frac{7\pi}{12} \cos \frac{\pi}{3} + \sin \frac{7\pi}{12} \sin \frac{\pi}{3} &= \cos\left(\frac{7\pi}{12} - \frac{\pi}{3}\right)\\ &= \cos\left(\frac{7\pi}{12} - \frac{4\pi}{12}\right)\\ &= \cos \frac{3\pi}{12}\\ &= \cos \frac{\pi}{4}\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

3. Textbook questions 9, 12, 13, 16, 17, 21, 24, 28, 30, 32, 33, 35, and 36 of "Practice," pp. 272 and 273

9. Method 1

$$\begin{aligned}\cos\frac{2\pi}{3} &= \cos\left(\frac{\pi}{3}+\frac{\pi}{3}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{3}-\sin\frac{\pi}{3}\sin\frac{\pi}{3}\\ &= \frac{1}{2}\left(\frac{1}{2}\right)-\frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{1}{4}-\frac{3}{4}\\ &= -\frac{2}{4}\\ &= -\frac{1}{2}\end{aligned}$$

Method 2

$$\begin{aligned}\cos\frac{2\pi}{3} &= \cos\left(\pi-\frac{\pi}{3}\right)\\ &= \cos\pi\cos\frac{\pi}{3}+\sin\pi\sin\frac{\pi}{3}\\ &= -1\left(\frac{1}{2}\right)-0\left(\frac{\sqrt{3}}{2}\right)\\ &= -\frac{1}{2}\end{aligned}$$

12. Method 1

$$\begin{aligned}\cos\frac{7\pi}{12} &= \cos\left(\frac{4\pi}{12}+\frac{3\pi}{12}\right)\\ &= \cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)-\frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{2}}{4}-\frac{\sqrt{3}\times\sqrt{2}}{4}\\ &= \frac{\sqrt{2}(1-\sqrt{3})}{4}\end{aligned}$$

Method 2

$$\begin{aligned}\cos\frac{7\pi}{12} &= \cos\left(\frac{9\pi}{12}-\frac{2\pi}{12}\right)\\ &= \cos\left(\frac{3\pi}{4}-\frac{\pi}{6}\right)\\ &= \cos\frac{3\pi}{4}\cos\frac{\pi}{6}+\sin\frac{3\pi}{4}\sin\frac{\pi}{6}\\ &= -\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= -\frac{\sqrt{2}\times\sqrt{3}}{4}+\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{2}}{4}-\frac{\sqrt{2}\times\sqrt{3}}{4}\\ &= \frac{\sqrt{2}(1-\sqrt{3})}{4}\end{aligned}$$

Section 2: Activity 5 (continued)

13. Method 1

$$\begin{aligned}\tan 105^\circ &= \tan\left(60^\circ + 45^\circ\right)\\ &= \frac{\tan 60^\circ + 45^\circ}{1 - \tan 60^\circ \tan 45^\circ}\\ &= \frac{\sqrt{3}+1}{1-\sqrt{3}\times 1}\\ &= \frac{\sqrt{3}+1}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}\\ &= \frac{\sqrt{3}+3+1+\sqrt{3}}{-2}\\ &= \frac{4+2\sqrt{3}}{-2}\\ &= -2-\sqrt{3}\end{aligned}$$

Method 2

$$\begin{aligned}\tan 105^\circ &= \tan\left(135^\circ - 30^\circ\right)\\ &= \frac{\tan 135^\circ - \tan 30^\circ}{1 + \tan 135^\circ \tan 30^\circ}\\ &= \frac{-1-\frac{\sqrt{3}}{3}}{1+(-1)\left(\frac{\sqrt{3}}{3}\right)}\\ &= \frac{-\frac{3+\sqrt{3}}{\cancel{3}}}{\frac{3-\sqrt{3}}{\cancel{3}}}\\ &= -\frac{3+\sqrt{3}}{3-\sqrt{3}} \times \frac{3+\sqrt{3}}{3+\sqrt{3}}\\ &= -\frac{9+6\sqrt{3}+3}{9-3}\\ &= -\frac{12+6\sqrt{3}}{6}\\ &= -2-\sqrt{3}\end{aligned}$$

16. Method 1

$$\begin{aligned}\cos\frac{5\pi}{6} &= \cos\left(\frac{3\pi}{6}+\frac{2\pi}{6}\right)\\ &= \cos\left(\frac{\pi}{2}+\frac{\pi}{3}\right)\\ &= \cos\frac{\pi}{2}\cos\frac{\pi}{3} - \sin\frac{\pi}{2}\sin\frac{\pi}{3}\\ &= 0\left(\frac{1}{2}\right) - 1\left(\frac{\sqrt{3}}{2}\right)\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

Method 2

$$\begin{aligned}\cos\frac{5\pi}{6} &= \cos\left(\pi - \frac{\pi}{6}\right)\\ &= \cos\pi\cos\frac{\pi}{6} + \sin\pi\sin\frac{\pi}{6}\\ &= -1\left(\frac{\sqrt{3}}{2}\right) + 0\left(\frac{1}{2}\right)\\ &= -\frac{\sqrt{3}}{2}\end{aligned}$$

17. $$\begin{aligned}2\sin\frac{\pi}{6}\cos\frac{\pi}{6} &= \sin\left(\overset{1}{\cancel{2}}\times\frac{\pi}{\underset{3}{\cancel{6}}}\right)\\ &= \sin\frac{\pi}{3}\end{aligned}$$

Recall: $2\sin A\cos A = \sin 2A$

Section 2: Activity 5 (continued)

21. $$\frac{\tan\pi - \tan\frac{\pi}{6}}{1+\tan\pi\tan\frac{\pi}{6}} = \tan\left(\pi - \frac{\pi}{6}\right)$$

$$= \tan\frac{5\pi}{6}$$

Recall: $\dfrac{\tan A - \tan B}{1+\tan A\tan B} = \tan(A-B)$

24. $$\cos\frac{\pi}{12}\cos\pi - \sin\frac{\pi}{12}\sin\pi = \cos\left(\frac{\pi}{12}+\pi\right)$$

$$= \cos\frac{13\pi}{12}$$

Recall: $\cos A\cos B - \sin A\sin B = \cos(A+B)$

28.

LS	RS
$\sin(270° - A)$	$-\cos A$
$= \sin 270°\cos A - \cos 270°\sin A$	
$= -1\times\cos A - 0\times\sin A$	
$= -\cos A$	

LS = RS

30.

LS	RS
$\cos(2\pi + A)$	$\cos A$
$= \cos 2\pi\cos A - \sin 2\pi\sin A$	
$= 1\times\cos A - 0\times\sin A$	
$= \cos A$	

LS = RS

32.

LS	RS
$\cos\left(\frac{3\pi}{2} - A\right)$	$-\sin A$
$= \cos\frac{3\pi}{2}\cos A + \sin\frac{3\pi}{2}\sin A$	
$= 0\times\cos A + (-1)\sin A$	
$= -\sin A$	

LS = RS

33. The equation $\cos 140° = \cos 60°\cos 80° - \sin 60°\sin 80°$ is true because $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

Section 2: Activity 5 (continued)

35. The equation $\sin \frac{5\pi}{12} = \sin \frac{\pi}{6} \cos \frac{\pi}{4} + \cos \frac{\pi}{6} \sin \frac{\pi}{4}$ is true because $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

$$\begin{aligned}\frac{\pi}{6}+\frac{\pi}{4}&=\frac{2\pi}{12}+\frac{3\pi}{12}\\&=\frac{5\pi}{12}\end{aligned}$$

36. This equation is false according to the sum identity for tangent, $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.

$$\tan 75° = \frac{\tan 35° + \tan 40°}{1 - \tan 35° \tan 40°}$$

4. Textbook questions 37, 39, 42, 45, 50, 52, and 53 of "Applications and Problem Solving," pp. 273 and 274

37. a. First, you need to find the values of $\cos A$ and $\sin B$. **Remember:** Sine and cosine are positive in the first quadrant.

$$\begin{aligned}\sin^2 A &= 1 - \cos^2 A\\ \left(\frac{3}{5}\right)^2 &= 1 - \cos^2 A\\ \frac{9}{25} &= 1 - \cos^2 A\\ \cos^2 A &= 1 - \frac{9}{25}\\ \cos^2 A &= \frac{16}{25}\\ \cos A &= \sqrt{\frac{16}{25}}\\ \cos A &= \frac{4}{5}\end{aligned}$$

$$\begin{aligned}\sin^2 B &= 1 - \cos^2 B\\ \sin^2 B &= 1 - \left(\frac{5}{13}\right)^2\\ \sin^2 B &= 1 - \frac{25}{169}\\ \sin^2 B &= \frac{144}{169}\\ \sin B &= \sqrt{\frac{144}{169}}\\ \sin B &= \frac{12}{13}\end{aligned}$$

Now, evaluate $\cos(A-B)$.

$$\begin{aligned}\cos(A-B) &= \cos A \cos B + \sin A \sin B\\ &= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{12}{13}\right)\\ &= \frac{20}{65} + \frac{36}{65}\\ &= \frac{56}{65}\end{aligned}$$

b. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \left(\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{12}{13}\right)$$

$$= \frac{15}{65} + \frac{48}{65}$$

$$= \frac{63}{65}$$

c. First, find the values of tan A and tan B.

$$\tan A = \frac{\sin A}{\cos A} \qquad \tan B = \frac{\sin B}{\cos B}$$

$$= \frac{\frac{3}{\cancel{5}}}{\frac{4}{\cancel{5}}} \qquad = \frac{\frac{12}{\cancel{13}}}{\frac{5}{\cancel{13}}}$$

$$= \frac{3}{4} \qquad = \frac{12}{5}$$

Now, evaluate $\tan(A+B)$.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \left(\frac{3}{4}\right)\left(\frac{12}{5}\right)}$$

$$= \frac{\frac{15+48}{\cancel{20}}}{\frac{20-36}{\cancel{20}}}$$

$$= \frac{63}{-16}$$

$$= -3\frac{15}{16}$$

Section 2: Activity 5 (continued)

39. Sine is negative in the fourth quadrant.

$$\therefore\ \sin^2 A = 1 - \cos^2 A$$

$$\sin^2 A = 1 - \left(\frac{12}{13}\right)^2$$

$$\sin^2 A = 1 - \frac{144}{169}$$

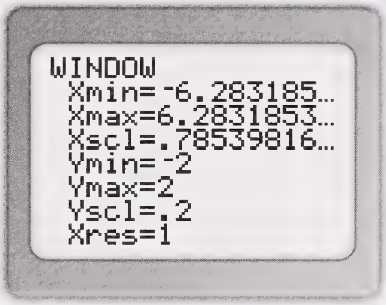
$$\sin^2 A = \frac{25}{169}$$

$$\sin A = -\sqrt{\frac{25}{169}}$$

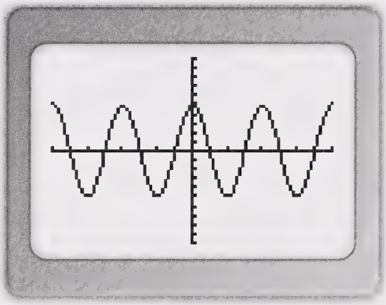
$$\sin A = -\frac{5}{13}$$

$$\begin{aligned}\therefore\ \sin 2A &= 2 \sin A \cos A \\ &= 2\left(-\frac{5}{13}\right)\left(\frac{12}{13}\right) \\ &= -\frac{120}{169}\end{aligned}$$

42. **a.**

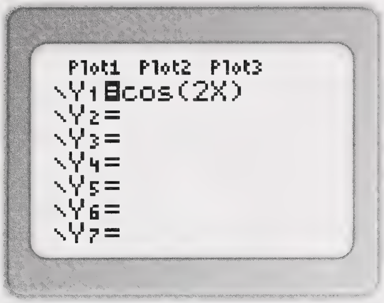


```
WINDOW
 Xmin=-6.283185...
 Xmax=6.2831853...
 Xscl=.78539816...
 Ymin=-2
 Ymax=2
 Yscl=.2
 Xres=1
```

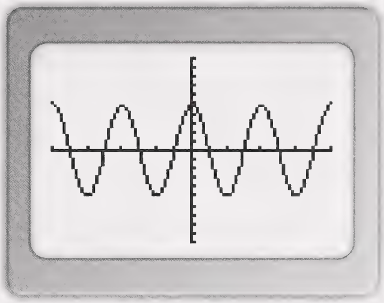


The function appears to have an amplitude of 1 and a period of π.

b. The graph suggests that $f(x) = \cos 2x$.

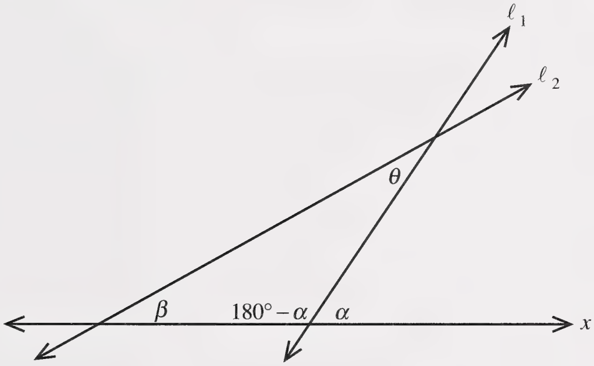


```
Plot1 Plot2 Plot3
\Y1=cos(2X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



c. $$\begin{aligned}(\cos x - \sin x)(\cos x + \sin x) &= \cos^2 x - \sin^2 x \\ &= \cos 2x\end{aligned}$$

45. The following diagram shows the angles described in the problem.



Because the sum of the angles of a triangle is 180°,

$$\begin{aligned}\theta + \beta + (180° - \alpha) &= 180° \\ \theta + \beta - \alpha &= 0 \\ \theta &= \alpha - \beta\end{aligned}$$

$$\begin{aligned}\therefore\ \tan\theta &= \tan(\alpha - \beta) \\ &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} \\ &= \frac{m_2 - m_1}{1 + m_2 m_1}\end{aligned}$$

Section 2: Activity 5 (continued)

50. **a.** A double-angle identity for cosine is $\cos 2A = 1 - 2\sin^2 A$.

$$\begin{aligned}
\cos 2A &= 1 - 2\sin^2 A \\
2\sin^2 A &= 1 - \cos 2A \\
\sin^2 A &= \frac{1-\cos 2A}{2} \\
\sin A &= \pm\sqrt{\frac{1-\cos 2A}{2}}
\end{aligned}$$

Substitute $\frac{x}{2}$ for A, and simplify.

$$\begin{aligned}
\sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos\left(2\times\frac{x}{2}\right)}{2}} \\
\sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}}
\end{aligned}$$

b. A double-angle identity for cosine is $\cos 2A = 2\cos^2 A - 1$.

$$\begin{aligned}
\cos 2A &= 2\cos^2 A - 1 \\
2\cos^2 A &= 1 + \cos 2A \\
\cos^2 A &= \frac{1+\cos 2A}{2} \\
\cos A &= \pm\sqrt{\frac{1+\cos 2A}{2}}
\end{aligned}$$

Substitute $\frac{x}{2}$ for A, and simplify.

$$\begin{aligned}
\cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos\left(2\times\frac{x}{2}\right)}{2}} \\
\cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}}
\end{aligned}$$

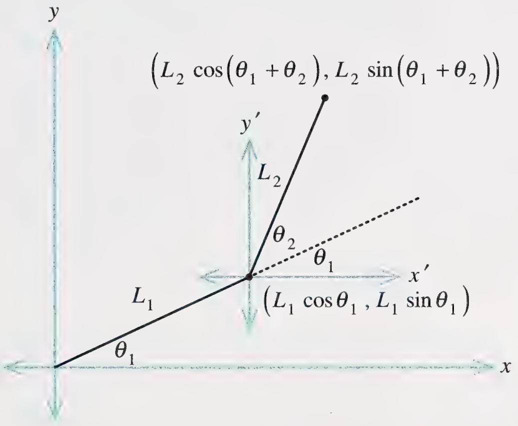
52.

$$\begin{aligned}(3x+3)(2y+2)&=12\\ [3(x+1)][2(y+1)]&=12\\ 6(x+1)(y+1)&=12\\ (x+1)(y+1)&=2\\ xy+x+y+1&=2\\ xy+x+y&=1\end{aligned}$$

Substitute tan A for x and tan B for y, and simplify.

$$\begin{aligned}\tan A\tan B+\tan A+\tan B&=1\\ \tan A+\tan B&=1-\tan A\tan B\\ \frac{\tan A+\tan B}{1-\tan A\tan B}&=1\\ \tan(A+B)&=1\\ A+B&=\tfrac{\pi}{4}+k\pi,\text{ where } k \text{ is any integer}\end{aligned}$$

53. a. The following diagram shows the robotic arm in simplified form.


y
$(L_2\cos(\theta_1+\theta_2), L_2\sin(\theta_1+\theta_2))$
y'
L_2
θ_2
θ_1
x'
L_1
$(L_1\cos\theta_1, L_1\sin\theta_1)$
θ_1
x

The coordinates of the point where the two sections of the arm meet are $(L_1\cos\theta_1, L_1\sin\theta_1)$ because every point on that line is of the form $(r\cos\theta_1, r\sin\theta_1)$, where r is the distance of the point from the origin.

Section 2: Activity 5 (continued)

Using the second coordinate system, located at the point where the arms meet, the coordinates of the end of the second arm are $\left(L_2 \cos(\theta_1 + \theta_2), L_2 \sin(\theta_1 + \theta_2)\right)$.

To find the coordinates of the end of the second arm in terms of the original coordinate system, the sum of these two coordinates is taken.

$$\left(L_2 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2), L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2)\right)$$

This is the same as the equations $x = L_1 \cos \theta_1 + L_2 \cos(\theta_1 + \theta_2)$ and $y = L_1 \sin \theta_1 + L_2 \sin(\theta_1 + \theta_2)$.

b. The gripper can reach at most 4 m, which is attained when the two arms are collinear.

c. The point $(3.3, 2.5)$ will not be reachable because it is more than 4 m from the origin.

$$\begin{aligned}\sqrt{(3.3)^2 + (2.5)^2} &= \sqrt{10.89 + 6.25} \\ &= \sqrt{17.14} \\ &> 4\end{aligned}$$

5. Textbook question 5 of "Using the Strategies," p. 281

5. The table that follows shows how the states of the locker doors as each child passes by. A 0 represents an open door, and a 1 represents a closed door. The doors that are changed by each child are shown in a different colour.

Locker	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
initial state	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
child 1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0		0	1	0	1	0	1	0
child 2	0	1	1	1	0		0	1		1	0	0	0	1	1	1	0		0	1		1	0	0	0
child 3	0	1	1	0	0	0	0	0	1	1	0	1	0	1	1	0	0	0	0	0	1	1	0	1	0
child 4	0	1	1	0		0	0	0	1	0	0	1	0	1	0	0	0	0	0	1	1	1	0	1	
child 5	0	1	1	0	1	1	0	0	1	0	0	0	0	1	0	0	0		0	1	1	1	0	0	1
child 6	0	1	1	0	1	1	1	0	1	0	0	0	0		0	0	0	1	0	1	0	1	0	0	1
child 7	0	1	1	0	1	1	1	1	1	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	1
child 8	0	1	1	0	1	1	1	1	0	0	0	0	0	0	0	1	0		0	1	0	1	0	1	1
child 9	0	1	1	0	1	1	1	1	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	1	1
child 10	0	1	1	0	1	1	1	1	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	1	1
child 11	0	1	1	0	1	1	1	1	0	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	1
child 12	0	1	1	0	1	1	1	1	0	1	1	1		0	0	1	0	0	0	0	0	0	0	0	1
child 13	0	1	1	0	1	1	1	1	0	1	1	1	1		0	1	0	0	0	0	0	0	0	0	1
child 14	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1
child 15	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	1
child 16	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	0		0	0	0	0	0	0	0	1
child 17	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	0	1		0	0	0	0	0	0	1
child 18	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	0	0	0	0	0	1
child 19	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	0	0	0	0	1
child 20	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1		0	0	0	1
child 21	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1		0	0	1
child 22	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	0	1
child 23	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1
child 24	0	1	1	0	1	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	

Lockers 1, 4, 9, 16, and 25 are open after the 24th child passes by.

Section 2: Follow-up Activities

Extra Help

1. **a.** $\sin^2 x + \cos^2 x = 1$

b. $\sin^2 x + \cos^2 x = 1$

c. $\sin 2x = 2 \sin x \cos x$

d. $\cos 2x = \cos^2 x - \sin^2 x$

e. $1 + \tan^2 x = \sec^2 x$

f. $\sin(A + B) = \sin A \cos B + \cos A \sin B$

g. $\cos(A + B) = \cos A \cos B - \sin A \sin B$

h. $1 + \tan^2 x = \sec^2 x$

Extra Help (continued)

2. **a.** $\sin^2 x - 1 = \cos^2 x$

b.
$$\begin{aligned} 10\cos^2 x - 10 &= 10\left(\cos^2 x - 1\right) \\ &= 10\sin^2 x \end{aligned}$$

c.
$$\begin{aligned} \frac{\sin 2x}{2\sin x} &= \frac{\cancel{2\sin x}\cos x}{\cancel{2\sin x}} \\ &= \cos x \end{aligned}$$

d.
$$\begin{aligned} \cos^2 x - \cos 2x &= \cos^2 x - \left(\cos^2 x - \sin^2 x\right) \\ &= \sin^2 x \end{aligned}$$

e.
$$\begin{aligned} \sin^2\frac{\pi}{7}\sec^2\frac{\pi}{7} + 1 &= \sin^2\frac{\pi}{7} \times \frac{1}{\cos^2\frac{\pi}{7}} + 1 \\ &= \frac{\sin^2\frac{\pi}{7}}{\cos^2\frac{\pi}{7}} + 1 \\ &= \tan^2\frac{\pi}{7} + 1 \\ &= \sec^2\frac{\pi}{7} \end{aligned}$$

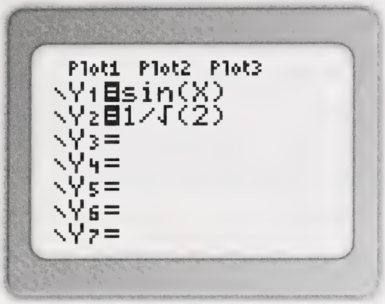
f.
$$\begin{aligned} \sin\frac{7\pi}{12} &= \sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right) \\ &= \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) \\ &= \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3} \\ &= \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{2} \times \sqrt{3}}{4} \\ &= \frac{\sqrt{2}\left(1 + \sqrt{3}\right)}{4} \end{aligned}$$

g.
$$\begin{aligned} \cos 75^\circ &= \cos\left(30^\circ + 45^\circ\right) \\ &= \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ \\ &= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{3} \times \sqrt{2}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{2}\left(\sqrt{3} - 1\right)}{4} \end{aligned}$$

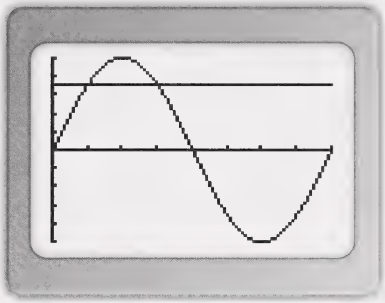
h.
$$\begin{aligned} 1 - \sec^2 x &= 1 - \left(1 + \tan^2 x\right) \\ &= -\tan^2 x \end{aligned}$$

3. Textbook questions 7, 12, 20, 24.a., 24.b., and 25 of "Review," p. 278

7. a.



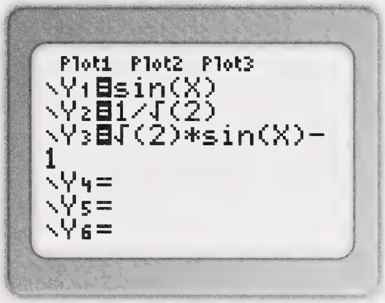
```
Plot1 Plot2 Plot3
\Y1=sin(X)
\Y2=1/√(2)
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



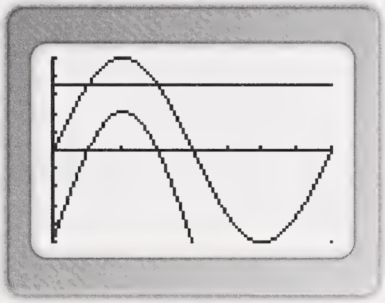
b. The graphs show two solutions to $\sin x = \frac{1}{\sqrt{2}}$. These solutions are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

The general solution is found by adding multiples of the period of the function to all solutions found in a single period of the function. Because the period is 2π, the general solutions are $x = \frac{\pi}{4} + 2k\pi$ and $x = \frac{3\pi}{4} + 2k\pi$, where k is any integer.

c. The zeros of the equation $\sqrt{2}\sin x - 1 = 0$ and the solutions to $\sin x = \frac{1}{\sqrt{2}}$ are the same values. This can be checked by graphing this new equation, $y = \sqrt{2}\sin x - 1$, along with the original pair of equations.



```
Plot1 Plot2 Plot3
\Y1=sin(X)
\Y2=1/√(2)
\Y3=√(2)*sin(X)-
1
\Y4=
\Y5=
\Y6=
```



Note that the new graph crosses the x-axis at the same x-values as the points where the other two graphs intersect.

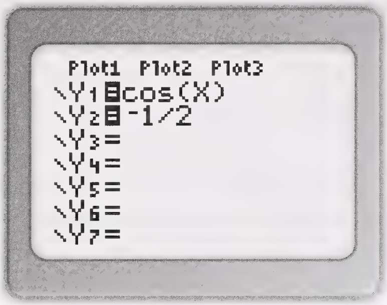
Extra Help (continued)

12. $2 \cos 2x + 1 = 0$

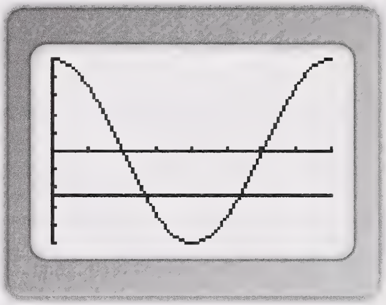
$$2 \cos 2x = -1$$

$$\cos 2x = -\frac{1}{2}$$

Graph $y = \cos x$ as Y_1 and $y = -\frac{1}{2}$ as Y_2.



```
Plot1 Plot2 Plot3
\Y1■cos(X)
\Y2■-1/2
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



The two graphs intersect at $x = 120°$ and $x = 240°$.

Because $y = \cos 2x$ is a horizontal compression of $y = \cos x$ by a factor of 2, the solution to $\cos 2x = -\frac{1}{2}$ in its first period are $x = \frac{120°}{2}$ or $60°$ and $x = \frac{240°}{2} = 120°$.

Because $y = \cos 2x$ has two periods in the interval $0 \le x < 360°$, the solutions in the second period are $x = 60° + 180°$ or $240°$ and $x = 120° + 180°$ or $300°$.

Therefore, the solutions to the equation $2 \cos 2x + 1 = 0$ in the interval $0 \le x < 360°$ are 60°, 120°, 240°, and 300°.

20. $\sin^2 2\theta + \sin 2\theta = 0$

$\sin 2\theta(\sin 2\theta + 1) = 0$

$\therefore \sin 2\theta = 0$ or $\sin 2\theta + 1 = 0$

$2\theta = 0$ and π $\quad$ $\sin 2\theta = -1$

$\theta = 0$ and $\frac{\pi}{2}$ $\quad$ $2\theta = \frac{3\pi}{2}$

$\theta = \frac{3\pi}{4}$

Because $y = \sin 2\theta$ is $y = \sin\theta$ compressed horizontally by a factor of 2, $y = \sin 2\theta$ will have two periods in the interval $0 \le \theta < 2\pi$. Therefore, in the second period, the solutions are $\theta = 0 + \pi$ or $\pi, \theta = \frac{\pi}{2} + \pi$ or $\frac{3\pi}{2}$, and $\theta = \frac{3\pi}{4} + \pi$ or $\frac{7\pi}{4}$.

The solutions to $\sin^2 2\theta + \sin 2\theta = 0$ in the interval $0 \le \theta < 2\pi$ are $0, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}$, and $\frac{7\pi}{4}$.

The general solution is found by adding multiples of the period, π, to the solutions from the first period. Therefore, the general solutions are $\theta = k\pi, \theta = \frac{\pi}{2} + k\pi$, and $\theta = \frac{3\pi}{4} + k\pi$, where k is any interger.

These solutions can be combined as $\theta = \frac{k\pi}{2}$ and $\theta = \frac{3\pi}{4} + k\pi$, where k is any integer.

24. **a.** $4\sin^2 x - 3\sin x - 1 = 0$

$(4\sin x + 1)(\sin x - 1) = 0$

$\therefore 4\sin x + 1 = 0$ or $\sin x - 1 = 0$

$4\sin x = -1$ $\quad$ $\sin x = 1$

$\sin x = -\frac{1}{4}$ $\quad$ $x = \frac{\pi}{2}$

$x \doteq 3.4$ and 6.0 $\quad$ $\doteq 1.6$

The solutions to $4\sin^2 x - 3\sin x - 1 = 0$ in the interval $0 \le x \le 2\pi$ are approximately 1.6, 3.4, and 6.0.

Extra Help (continued)

b. $$3\cos^2 x - \cos x = 0$$
$$(\cos x)(3\cos x - 1) = 0$$

$\therefore\ \cos x = 0$ or $3\cos x - 1 = 0$

$$x = \frac{\pi}{2} \text{ and } \frac{3\pi}{2}$$
$$\doteq 1.6 \text{ and } 4.7$$

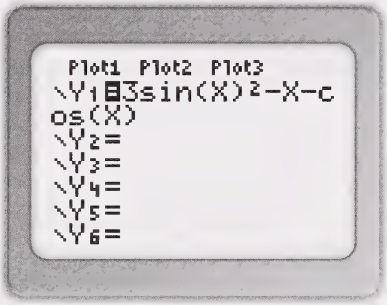
$$3\cos x = 1$$
$$\cos x = \frac{1}{3}$$
$$x \doteq 1.2 \text{ and } 5.1$$

The solutions to $3\cos^2 x - \cos x = 0$ in the interval $0 \le x < 2\pi$ are approximately 1.2, 1.6, 4.7, and 5.1.

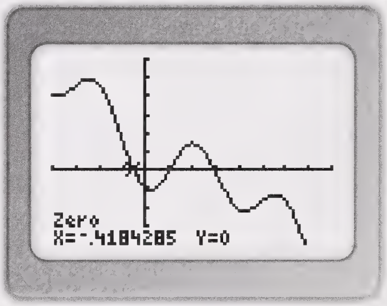
25. a. There are three solutions to the equation $3\sin^2 x = x + \cos x$.

b. Graph the function $y = 3\sin^2 x - x - \cos x$ to solve the equation $3\sin^2 x = x + \cos x$.

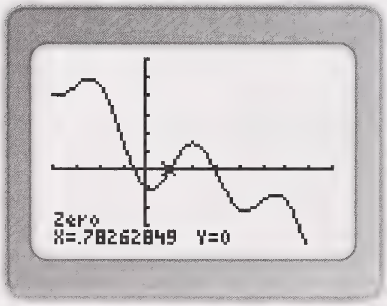
Use the Zero feature on your graphing calculator to find the solution to $3\sin^2 x = x + \cos x$.



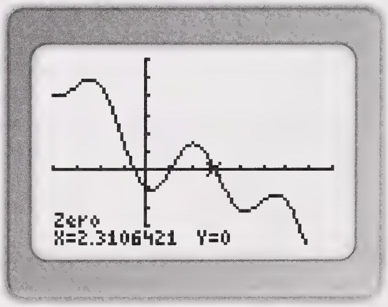
```
 Plot1 Plot2 Plot3
\Y1■3sin(X)²-X-c
os(X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



```
Zero
X=-.4184285  Y=0
```



```
Zero
X=.78262849  Y=0
```

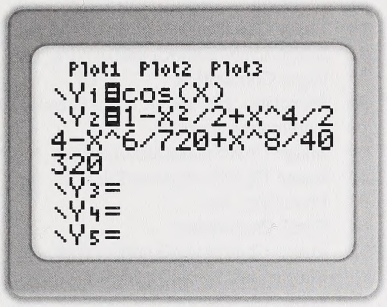


```
Zero
X=2.3106421  Y=0
```

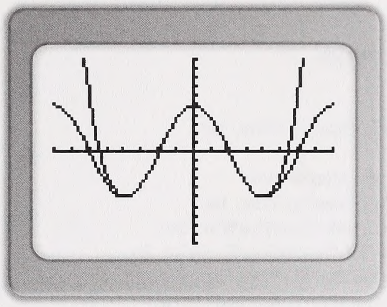
The solutions are approximately -0.418, 0.783, and 2.311.

Enrichment

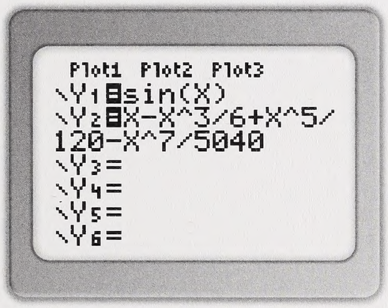
1. The hint suggests using $6\times5\times4\times3\times2\times1$ and $8\times7\times6\times5\times4\times3\times2\times1$ as the denominators for the next two terms in the function. $\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40\,320}\right)$ is a good approximation for cosine from $-\pi\le x\le\pi$.



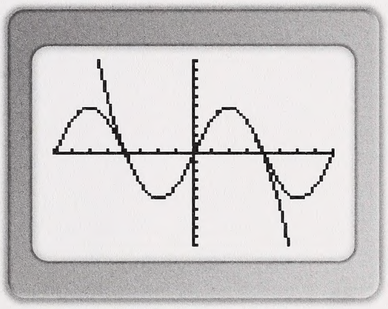
```
Plot1 Plot2 Plot3
\Y1=cos(X)
\Y2=1-X²/2+X^4/2
4-X^6/720+X^8/40
320
\Y3=
\Y4=
\Y5=
```



2. Because sine is an odd function, the powers used should be odd. $\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}\right)$ would be a good approximation of sine around zero.



```
Plot1 Plot2 Plot3
\Y1=sin(X)
\Y2=X-X^3/6+X^5/
120-X^7/5040
\Y3=
\Y4=
\Y5=
\Y6=
```



3. Experimentation will give an exponent value around 34 before the pink graph covers the blue graph. It will take an exponent value around 42 before the data points shown in columns A and B begin to match to four decimal places.

Credits

Some clip art graphics are commercially owned.

Welcome Page
PhotoDisc, Inc.

Page	
1	PhotoDisc, Inc.
2–3	**collage:** PhotoDisc, Inc.
4	PhotoDisc, Inc.
6	PhotoDisc, Inc.
8	Image Club/StudioGear/EyeWire, Inc.
12	PhotoDisc, Inc.
15	Nova Development Corporation
17	Image Club/StudioGear/EyeWire, Inc.
26	**top:** Image Club/StudioGear/EyeWire, Inc. **bottom:** Image Club/StudioGear/EyeWire, Inc.
33	Image Club/StudioGear/EyeWire, Inc.
34	**collage:** PhotoDisc, Inc. (radios and stereo); LTB (CD player)
38	PhotoDisc, Inc.
42	PhotoDisc, Inc.
43	PhotoDisc, Inc.
49	Image Club/StudioGear/EyeWire, Inc.
51	Image Club/StudioGear/EyeWire, Inc.
55	Corel Corporation
58	PhotoDisc, Inc. (boy); Nova Development Corporation (cake)
59	Image Club/StudioGear/EyeWire, Inc.
62	Image Club/StudioGear/EyeWire, Inc.
63	PhotoDisc, Inc.
64	Corel Corporation
65	Image Club/StudioGear/EyeWire, Inc.
73	Image Club/StudioGear/EyeWire, Inc.
75	Image Club/StudioGear/EyeWire, Inc.
76	PhotoDisc, Inc.
81	**bottom:** EyeWire, Inc.
83	Image Club/StudioGear/EyeWire, Inc.
85	**bottom:** Image Club/StudioGear/EyeWire, Inc.

Pure Mathematics 30
Student Module Booklet
Module 5

LRDC

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